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GERHARD N. WOLLAN, *EDITOR*

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RECOLLECTIONS AND REFLECTIONS

R. L. WILDER, University of California, Santa Barbara

According to a well-known aphorism (of Pennsylvania Dutch origin, I presume):

"We grow too soon old,
Und too late shmart."

Possibly as it applies to one's diet and other practical affairs of life, it embodies a great deal of truth. But it does seem to run counter to the present folklore of mathematics, which is frequently stated in the form,

"Mathematics is a young man's game."

The latter seems to imply that the successful mathematician must be "shmart" while young and that he will have declining powers as he grows older. This is probably true, in general, although such classical examples as Hadamard, Hilbert, Mordell and Weierstrass may cast doubt on it. I believe that Weierstrass was 40 years of age by the time he began to attract the notice of the mathematical world.

On the other hand, experience does teach, and the longer one lives, the more he learns about topics that he wishes he had known earlier. Moreover, he comes to realize that he has seen history in the making. Events that at the time seemed quite ordinary, later turn out to have been of great significance. And sometimes persons that one has known and taken for granted during daily contacts ultimately achieve sufficient recognition to warrant them prominent places in the history of mathematics. A few years ago, I was a bit startled when my wife told me that one of my students had told her that one of the things he liked best about a course he was taking with me was the incidental remarks I frequently made about people I had known and who were responsible for material being studied; it sounded as though I had been indulging in "name-dropping." Actually, I had been trying to convey some idea to the students of how mathematics is created, and that the creators are human beings like themselves.

For example, anyone who regularly attended Eastern mathematical meetings during the years surrounding World War II would very likely have become acquainted with Felix Bernstein, who was at this time living in the United States and doing biometric work for a medical group. His tales about the difficulties he encountered in obtaining materials for fashioning laboratory gadgets revealed an aptitude for invention quite unexpected in one who is probably most commonly remembered in connection with the famous equivalence theorem which bears his name. Students learning about the theorem for the first time are always fascinated to hear about this "non-abstract" side of Bernstein's character.

Of course, there may be times when one might wish not to reveal such chance acquaintanceships. Not long ago when inquiring for a prescription in my name at my neighborhood drugstore, I was startled by a new clerk's greeting: "You're the man who knew Einstein!" It reminded me of a book written by the late Sinclair Lewis entitled, "The man who knew Coolidge." Upon asking how he found that interesting bit of information, I was informed that some time before, the 12 year-old son of the proprietor, who had somehow learned I was a mathematician, asked me if I ever knew Einstein. Evidently my affirmative reply created an aura which I can now feel every time I enter the store!

Attending a meeting such as the present one, back in the twenties, was vastly different in point of number attending. This was true especially for sectional meetings of the American Mathematical Society at that time. Meetings of the midwest section, usually held in Chicago in April, were quite "cozy" affairs. There was always a banquet — they hadn't been declared impractical at that time — and frequently it was held at one long table, seating, I would guess, around 50 people at most. Almost invariably there would be found at the table such men as E. H. Moore, L. E. Dickson, E. B. VanVleck (after whom the mathematics building at the University of Wisconsin is named) and others that I now realize played notable roles in early American mathematics as well as in the founding of the Association.

As I compare the size of meetings — even the annual ones — then and now, I am not convinced that the present lack of jobs, and talk of holding down the number of Ph. D.'s, are due entirely to economic reasons; unless, indeed, the economic causes are themselves a byproduct of what seems inevitable. According to studies made by scholars who study the growth of science, such as Derek Price of Yale [**4**], all sciences seem to have been following the normal growth curve; i. e., slow start, eventual pickup with an exponential type growth, and then a tendency to level off and approach a saturation level. After all, we live on a finite earth which can support only a finite number of people. Of these only a finite number can become mathematicians and/or create a demand for what mathematicians can do. It seems not unlikely that the present conditions only presage what we could only expect to happen in the near future, no matter what the general economic and social conditions may be.

When I began my study of mathematics, American mathematics was at what, looking back, I now consider a generally low point. The demand for the so-called "practical" was slowly pressuring the textbook writer to turn his products into virtual handbooks for engineers. At that time there was not the demand from the natural sciences that one finds today, and precious little demand from the social sciences, which were not so well developed as they are today. As a consequence, the mathematics students in the university undergraduate colleges were mostly from engineering. The derisive term "handbook boys" for the engineers of those days was not unjustified. I recall how Granville's calculus [3], published in 1904, was modified by Longley and others in order to meet the demand for less theory; the original Granville had something of solidity to it, as well as an unusual collection of problems. It contained, for instance, de la Vallee-Poussin's proof of the existence of e, the natural base of logarithms; this was one of the first casualties, I need hardly say. Granville's exercises, however, were to be found in most of the new textbooks on calculus for many years, while the texts themselves degenerated into a series of prescriptions for doing the exercises.

Similar criticisms could be made of the other subjects taught. Geometry was represented mostly by synthetic, projective, and descriptive geometries, and algebra by solution of equations and matrix theory; a course in group theory was a rarity. In function theory the usual topics were from classical analysis; the "new" Lebesgue integration was given passing recognition as a mathematical oddity not likely ever to be of much use; probability was taught mostly in actuarial classes. The theory of analytic functions as taught in the complex variable courses was given as a "monumental achievement of the human mind," to use the words of one of my teachers, but unlikely ever to be of much use.

Fortunately, throughout this period, a sturdy body of men, devoted to their art (as they frequently called it), persisted in following up the new concepts emanating from the universities of France, Germany and Poland. During the twenties, more of the universities began to award Ph. D.'s in pure mathematics. I am afraid the recipients of these degrees, however, still felt the pressure for the "practical." Those of you who have read N. Wiener's autobiography [**6**] will recall his feelings regarding his discovery, independently of Banach, of what were later called "Banach spaces." He wrote that "Banach space did not seem to have the physical and mathematical texture I wanted for a theory on which I was to stake a large part of my future reputation. . . . the theory seemed to me to contain for the immediate future nothing but some decades of rather formal and thin work." His feelings changed later, however, for in 1955, when writing the autobiography, he felt that "Nowadays it seems to me that some aspects of theory of Banach space are taking on a sufficiently rich texture and have been endowed with a sufficiently unobvious body of theorems to come closer to satisfying me in these respects."

Probably many of us who took our degrees in this country during the '20's felt the same way. I remember that the general spaces of topology were beginning to receive some attention at that time, but much as in the case of Wiener and his "Banach spaces", I felt that they were too "far out in left field," so to speak. Later developments have proved me wrong, of course; today a course in "general topology" is fundamental to work in modern analysis. In the American cultural climate of the '20's, few of us realized that the more abstract and general concepts were likely to prove the more important 50 years hence, both in mathematics and science. And it took genuine devotion to do research in the new ideas while carrying teaching loads of 12, 16 and even 18 hours per week, as well as performing those advisory, committee and administrative functions with which many of you are, I am sure, only too familiar.

I suppose most of us date the change in this state of affairs to the second World War. However, we must not forget that earlier we received a few boosts from the theoretical physicists. The revolution in physics evidenced by such theories as relativity and quantum mechanics showed that what had been considered pure abstract mathematics could be applied. Einstein himself had a high regard for basic mathematics; I often quote his statement to the effect that, "The progress entailed by axiomatics consists in the sharp separation of the logical form and the . . . intuitive contents . . . . To this interpretation of geometry I attach great importance because if I had not been acquainted with it, I would never have been able to develop the theory of relativity" [1]. And A. N. Whitehead's analogous statement in 1925 that, "The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact" [5].

Of even greater significance, however, was the opening of the Institute for Advanced Study in 1933. Fortunately the Bambergers, who founded the Institute, had the rare good fortune of finding a man whose knowledge of universities and their ways was second to none, viz., Abraham Flexner. The Institute was really his brain-child. He both engineered its inception and served as its director for the first six years of its existence. The story of how Flexner set up and conducted the Institute is told in Chapters 23 and 24 of his autobiography [2]. In particular, he states that after deciding to commence the Institute with emphasis on science: "The decision not to begin with the physical or biological sciences had become stronger, inasmuch as they were so well and widely carried on. Moreover, they do not lie at the very foundation of modern science. That foundation is mathematics. Mathematics is the servant of disciplines, antecedent, on the one hand, to science, and, on the other, to philosophy and economics and thus to the other social disciplines. With all its abstractness and indifference, both pure and applied scientific philosophic progress of recent years has been closely bound up with the new types and methods of sheer mathematical thinking." (*loc. cit.,* p. 243.)

The last part of this statement is even more true today than it was when Flexner wrote it. And the uses of mathematics during the war and especially the post-Sputnik era played a large part in making it so. I believe that the past 30 years constitute a period in which modern mathematics has reached full flower. Awareness that mathematics is really a science of structures and relations dominates research. It may seem strange that a science whose beginnings we can trace to at least 3,000 B.C. should only within the past century appear to have reached maturity. However, this overlooks the fact that "maturity" is really a function of the era. I warrant that after Greek mathematics had been formalized in the manner found in Euclid's *Elements*, the Hellenes considered that mathematics had reached maturity. And for that time it had; from our standpoint it had not. So while we may feel that now mathematics has finally matured, a later era in a different culture may feel differently. At any rate, during the past 30 years we have learned not only what we are trying to do, but we have also learned a great deal about our limitations — what we cannot do.

Also, with the help of Sputnik, we have taken steps toward unifying our profession, establishing liaison between the elementary and intermediate grade teachers and college teachers — a relationship which, unfortunately, we seem in danger of losing. I certainly do not delude myself that with the so-called "New Math" we have corrected all the shortcomings of elementary education in mathematics. I believe we have, however, hastened the movement of college mathematics down into the high schools and of the high school to the lower grades — a movement which was actually under way long before the war, to be sure, and which appears to be a natural evolution within our mathematical culture.

As I have grown older, I have noticed how the calibre of our best students, both undergraduate and graduate, has risen markedly. Also, how much greater power and control over his problems today's mathematician has than we used to have. These things do not signify that we have been attracting better minds into the profession — this would be a naive assumption — although it can be true to some extent. It does signify, I am convinced, that (1) we are profiting from the greater power accruing from wider points of view of a structural nature — a movement that began as long ago as Abel and Galois, but has only in the present century gained momentum; (2) we have been doing a better job teaching, not due simply to a better breed of teachers but to the existence of improved materials and methods; and (3) we are reaping the advantages of our students learning the true nature of mathematics at an earlier age.

Just how much we owe to the lower teaching loads in the universities since Sputnik and to the founding of the grants system can only be conjectured. I have always believed that anyone who wanted to do mathematics would do it even under the most adverse circumstances. When we were teaching 12 and 15 hours a week in the universities, a lot of good research was done. Incidentally, I must deplore the tenor of a recent article in the education department of the MONTHLY, which accentuated the so-called gap between teaching and research. According to the author, the universities do a poor job of teaching, although they do good research, while the colleges cannot do research with their heavy teaching loads, but do an excellent job of teaching. I believe such conclusions are not justified. I think that the universities and colleges *both* do an excellent job of teaching, generally speaking. Unfortunately, if a teacher in the colleges does a piece of research that attracts attention, he is soon snapped up by a research-oriented university. But enough of this.

What about the future? Certainly "educated" guesses about the future should be guided by the past. As for myself, I think that one of the most important lessons I have learned is that it is dangerous to predict what kind of mathematics will be most important in the future. Of all guides, however, the least reliable is practical utility. Let me quote further from Flexner (*loc. cit.*):

"[Mathematics] has uses, to be sure, as all the higher activities of the human mind have uses, if the word *use* is broadly and deeply understood. But its devotees are singularly unconcerned with use, most of all with immediate use .... Nothing is more likely to defeat itself, nothing is on the whole less productive, in the long run, than immediacy in the realm of research, reflection, and contemplation. The men who have moved the world have usually been men who have followed the will-o'-the-wisp of their own intellectual and spiritual curiosity ....

".... Mathematicians deal with intellectual concepts that they follow out for their own sake regardless of their possible usefulness, but, through this very freedom to pursue the apparently useless, they stimulate scientists, philosophers, economists, poets, and musicians, though without being at all conscious of any need or responsibility to do so." You see, incidentally, how wise were the Bambergers, merchants by profession, in their choice of the Institute's first director.

The first faculty Flexner chose consisted of O. Veblen, J. W. Alexander, J. von Neumann, Kurt Gödel, Hermann Weyl and A. Einstein. The first four, at that time, would have been classed as pure mathematicians. Weyl, while both pure mathematician and philosopher, had some interest in physics, while Einstein was a theoretical physicist. But as their ideas developed, all of these men, with the possible exception of Alexander, gave thought to applications. This was so particularly of von Neumann, whose interest in quantum mechanics, economics and computing are now well known; the general public knows him, if at all, only as a member of the Atomic Energy Commission and one who helped with the development of the atomic bomb. Also, Gödel, who is probably known to mathematicians chiefly for his theorems on incompleteness and the consistency of the continuum hypothesis, published three papers in 1949–50 concerning relativity theory — one of which, on the *Einstein Field Equations*, appeared in the *Review of Modern Physics*. Nevertheless, it was the genuine so-called core of mathematics that nurtured all of these men; even Einstein, whom I quoted earlier, paid it tribute as being of central importance to his way of thinking.

Now if these observations mean anything for the future, they certainly indicate that we must never neglect the core of mathematics. Applications seem to take care of themselves. One has only to look at the spread covered by mathematical organizations and journals today to realize that *all* aspects of mathematical activity, from pure to applied, are being covered. This is as it should be, and I hope it can continue.

At the same time, we must realize that we are reaching a saturation point at which neither the number of mathematicians nor the production rate of mathematics can be expected to increase further. How soon this will occur one cannot predict; but I think it is realistic to keep in mind that it will inevitably occur, and unfortunately the present economic situation may only hasten it. Perhaps we should so train our mathematics majors that they can assume tasks other than teaching or doing technical mathematics in industry or government. But that is another story.

A luncheon talk given to the March 11, 1972 meeting of the Southern California Section of the Association at California Institute of Technology.

Acknowledgment. The quotations from "Abraham Flexner, An Autobiography", by A. Flexner (1960), have been reprinted with permission from Simon and Schuster, Inc., New York.

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FINITE SETS

JEAN E. RUBIN, Purdue University

By the time a student finishes elementary school he "knows" what a finite set is. If using $1, 2, 3, \cdots$ he can count the elements in the set and get to the end the set is finite. While this concept seems rather simple, its further study is surprisingly interesting, is readily accessible to the college mathematics major, and serves as an introduction to a serious study of the methods and ideas of set theory.

Our discussion takes place within a standard system of set theory which we describe informally. The axiom which enables us to determine when two sets are equal is called the Axiom of Extensionality: two sets are equal if and only if (iff) they have the same elements, that is $x = y$ iff $(\forall u)\,(u \in x$ iff $u \in y)$. "$(\forall u)$" is an abbreviation for "for all u". In other words, two sets are equal iff each is a subset of the other. A set x is a subset of y, $x \subseteq y$, iff $(\forall u)\,(u \in x \to u \in y)$. The symbol "$\to$" is used for "if $\cdots$ then". We assume that all the sets we talk about have been constructed in some "known" way and that if x and y are sets so is their union, $x \cup y$; their intersection, $x \cap y$; their difference, $x \sim y$; and their unordered pair, $\{x, y\}$, the set whose only elements are x and y. If $x = y$ we obtain the unit set $\{x\}$. We also assume that if x is a set then the collection of all subsets of x, $\mathscr{P}(x)$, is a set, and if F is a function whose domain is a set then the range of F is a set.

We define sets which we shall call *natural numbers* as follows:

$$
\begin{aligned}
0 &= \varnothing \text{ (the empty set)}\\
1 &= \{0\}\\
2 &= \{0, 1\}\\
3 &= \{0, 1, 2\}\\
&\text{etc.}
\end{aligned}
$$

Each natural number is the set of all natural numbers which precede it. The one axiom we assume which isn't as constructive as we might like is the so-called *Axiom of Infinity*: there exists a set x such that $\varnothing \in x$ and for all u, if $u \in x$ then $u \cup \{u\} \in x$. A set x with this property is called an *inductive set* or a *successor set*, and "$u \cup \{u\}$" is denoted by "u^+" or "$u + 1$" and is called the *successor* of u. The axiom of infinity ensures that there is an infinite set and, in particular, that the collection of natural numbers, which is the intersection of all inductive sets (hence the "smallest" inductive set) is a set. (We assume that $u \notin u$ so that for every u, u is a proper subset of $u \cup \{u\}$. See [2], p. 81.) We use the symbol "N" for the set of natural numbers. A set of axioms for the system described here can be found in [2], but this informal description will be adequate for our purposes.

To express our intuitive notion of finiteness carefully we define the notion of two sets having the same number of elements. We say that two sets x and y are *equipollent* or have the *same cardinality*, $x \approx y$, if there is a 1-1 function mapping x onto y. Such a function is called a *bijection*. Symbolically, we write $F: x \approx y$ if F is a bijection with domain x and range y. The notion of a finite set can be expressed in set theoretical terms as follows:

DEFINITION 1. *A set x is finite if x is equipollent to some natural number. Otherwise, the set x is said to be infinite.*

The principal tool used to derive properties of finite sets is *mathematical induction*: if $x \subseteq N$ such that $0 \in x$ and for every n, whenever $n \in x$, $n^+ \in x$, then $x = N$. Mathematical induction is clearly just another way of saying that N is the "smallest" inductive set. Another useful formulation of mathematical induction is the *least number principle*: every nonempty set of natural numbers has a smallest element. For a discussion of mathematical induction and properties of finite sets, see, for example, [**2**], Chapter 4.

Using mathematical induction it is easy to prove the following properties of finite sets:

(I) If x and y are finite so is $x \cup y$.

(II) Every subset of a finite set is finite.

(III) If x is finite so is $x \cap y$ and $x \sim y$ for every y.

(IV) If x is finite so is $\mathscr{P}(x)$, the set of all subsets of x.

(V) If x is finite, $y \subseteq x$, and $x \approx y$, then $x = y$.

(VI) Every nonempty finite set of natural numbers has a largest element.

It follows from (V) that if two natural numbers are equipollent then they are identical because for any pair of natural numbers, one is a subset of the other. Therefore, if a set x is equipollent to a natural number n, the natural number is unique, so it is correct to say that "x has n elements".

In this paragraph we give definitions of a *partial ordering*, a *linear ordering*, a *well-ordering*, and *maximal*, *minimal*, *largest*, and *smallest* elements of a set. These notions are used in what follows.

(1) A relation R *partially orders* a set x if

(i) R is reflexive on x $[(\forall u \in x) uRu]$.

(ii) R is antisymmetric on x $[(\forall u, v \in x)\ (uRv \text{ and } vRu \rightarrow u = v)]$.

(iii) R is transitive on x $[(\forall u, v, w \in x)\ (uRv \text{ and } vRw \rightarrow uRw)]$.

The inclusion relation, $\subseteq$, on any set is an example of a partial ordering.

(2) We distinguish between maximal and largest elements of a set (with respect to some relation) and similarly between minimal and smallest elements. A maximal element has the property that there is nothing larger than it while a largest element has the property that it is larger than everything else. Similarly for minimal and smallest elements So, for example, if

$$x = \{\{1\}, \{2\}, \{1,3\}\}$$

then $\{1,3\}$ is maximal with respect to the subset relation since it is not a subset of any other set in x, but $\{1,3\}$ is not the largest element in the set because $\{2\}$ is not a subset of $\{1,3\}$. However, it is easy to see that a largest element is always a maximal element. We give the following formal definitions of these terms.

(2.1) u is an *R-maximal* element of x if $u \in x$ and $(\forall v \in x)\ (uRv \rightarrow vRu)$.

(2.2) u is an *R-minimal* element of x if $u \in x$ and $(\forall v \in x)\ (vRu \rightarrow uRv)$.

(2.3) u is an *R-largest* element of x if $u \in x$ and $(\forall v \in x)\ (vRu$ or $v = u)$.

(2.4) u is an *R-smallest* element of x if $u \in x$ and $(\forall v \in x)\ (uRv$ or $v = u)$.

(3) A relation R *linearly orders* a set x if

(i) R partially orders x.

(ii) R is connected on x $[(\forall u, v \in x)\ (uRv$ or $vRu)]$.

For example, the set of rational numbers is linearly ordered by $\leqq$.

(4) A relation R *well-orders* a set x if

(i) R linearly orders x.

(ii) Every nonempty subset of x has an R-smallest element.

The set of natural numbers is well-ordered by $\leqq$. Because of the way we have defined the natural numbers, the relation $\leqq$ on N is the same as the subset relation, $\subseteq$, and we use these symbols interchangeably when talking about N. Also, to be precise, when the terms "largest" and "smallest" were used previously in this paper, they should be replaced by "$\subseteq$-largest" and "$\subseteq$-smallest", respectively.

Since the notion of a finite set is intuitively so simple, it is aesthetically desirable to define it and derive its properties without using such a powerful tool as the axiom of infinity. Why should an infinite set be required for the study of finite sets? This same thought occurred to the Polish mathematician, Alfred Tarski, in the first quarter of the twentieth century, and he developed a theory of finite sets which did not depend on the axiom of infinity. Much of what follows can be found in his paper [**3**].

We give several alternative characterizations of a finite set.

THEOREM 1. *A set x is finite iff every nonempty subset of $\mathscr{P}(x)$ has a $\subseteq$-maximal (or $\subseteq$-minimal) element.*

Proof. First, suppose x is finite. Then it follows from property (V) of finite sets that $\mathscr{P}(x)$, the set of all subsets of x, is also finite. Suppose S is a nonempty subset of $\mathscr{P}(x)$. We want to prove that S has a $\subseteq$-maximal element. Since S is a subset of a finite set, S must be finite (property (II). Moreover, since $S \subseteq \mathscr{P}(x)$ every element of S is a subset of x. By hypothesis, x is finite, so by property (II), every subset of x is finite. Therefore, every element of S is finite. For each $u \in S$, let $|u|$ be the number of elements in u. Let y be the set of all $|u|$ such that $u \in S$. Symbolically, we write:

$$y = \{|u| : u \in S\}.$$

We have shown that y is a finite set of natural numbers. It follows from property (VI) that y must have a largest element. Call it m. Let v be an element of S such that $|v| = m$. This element v is a $\subseteq$-maximal element of S because if v were a proper subset of some element $w \in S$ then $m = |v| < |w|$ which contradicts the definition of m. There cannot be any $w \in S$ with $m < |w|$. In a similar way we can show that S must also have a $\subseteq$-minimal element.

Now, let us look at the converse. Suppose every nonempty subset of $\mathscr{P}(x)$ has a $\subseteq$-maximal element. Suppose also that x is not finite. We derive a contradiction by constructing a nonempty subset of $\mathscr{P}(x)$ which has no $\subseteq$-maximal element. It is left as an exercise to construct a nonempty subset of $\mathscr{P}(x)$ which has no $\subseteq$-minimal element. Let $E(x)$ be the set of all finite subsets of x:

$$E(x) = \{u : u \subseteq x \text{ and } |u| \in N\}.$$

We claim $E(x)$ has no $\subseteq$-maximal element. For suppose u is a $\subseteq$-maximal element of $E(x)$. Then, since u is finite and x is not we must have $x \sim u \neq \varnothing$. Let $a \in x \sim u$. The set $\{a\}$ is finite. Therefore, it follows from property (I) that $u \cup \{a\}$ is finite and by construction, $u \cup \{a\}$ is a subset of x. Consequently, $u \cup \{a\} \in E(x)$, and u is a proper subset of $u \cup \{a\}$. This contradicts the assumption that u was a $\subseteq$-maximal element of $E(x)$, so $E(x)$ cannot have a $\subseteq$-maximal element. We have derived the promised contradiction, so we conclude that x must be finite. This completes the proof of the theorem.

In our next theorem we use the symbol "R^{-1}" for the inverse of a relation R; that is,

$$aR^{-1}b \text{ iff } bRa.$$

THEOREM 2. *A set x is finite iff there is a relation R on x such that both R and R^{-1} well-order x.*

Proof. Suppose x is finite. Then, by Definition 1, there is a natural number n such that $x \approx n$. In other words, x has n elements, so we can represent the set x in the following way:

$$x = \{a_1, a_2, a_3, \cdots, a_n\}.$$

If $n = 0$ then $x = \varnothing$. Define a relation R on x such that for each $i, j = 1, 2, 3, \cdots, n$,

$$a_iRa_j \text{ iff } i \leqq j.$$

Then, if we use the fact that an R^{-1}-smallest element is the same as an R-largest element, it is easy to show that both R and R^{-1} well-order x.

Conversely, suppose there is a relation R on x such that both R and R^{-1} well-order x. Suppose also that x is not finite. We give an indirect proof by showing that if x is not finite and R well-orders x, then there is a subset of x which has no R^{-1}-smallest element. From the assumption that x is not finite and R well-orders x, it follows that for each finite subset u of x, $x \sim u \neq \varnothing$ and therefore must have an R-smallest element. Using mathematical induction we define a function F with domain N as follows:

$$F(0) = \text{the } R\text{-smallest element of } x$$

$$F(1) = \text{the } R\text{-smallest element of } x \sim \{F(0)\}$$

$$F(2) = \text{the } R\text{-smallest element of } x \sim \{F(0), F(1)\}$$

$$F(3) = \text{the } R\text{-smallest element of } x \sim \{F(0), F(1), F(2)\}$$

etc.

For a proof that this type of function exists, see, for example, [**2**], pp. 91–95.

Let y be the range of the function F. Now, $y \subseteq x$, but we claim that y has no R^{-1}-smallest element. For suppose s is an R^{-1}-smallest element of y. Then $s \in y$ so there is an $n \in N$ such that $s = F(n)$. By the definition of F,

$$F(n)\,R\,F(n+1) \text{ and } F(n) \neq F(n+1).$$

In other words,

$$F(n+1)\,R^{-1}s \text{ and } F(n+1) \neq s.$$

This contradicts the fact that s is the R^{-1}-smallest element of y. Thus y does not have any R^{-1}-smallest element. This implies that R^{-1} does not well-order x, a contradiction. Therefore, we conclude that x is finite.

In the proof of our next theorem we use the well-known mathematical technique of reducing the problem to a problem which already has been solved.

THEOREM 3. *A set x is finite iff x can be linearly ordered and every linear ordering of x is a well-ordering.*

Proof. The proof that if x is finite then x can be linearly ordered is the same as the first part of the proof of Theorem 2. So suppose x is finite and R is a relation which linearly orders x. We wish to show that R well-orders x. We again give an indirect proof. Since x is finite, it follows from (II) that every subset of x is also finite. Thus, if R does not well-order x then there must be some nonempty finite subset of x which has no R-smallest element. To show this is impossible we apply the least number principle. Let u be a nonempty subset of x with the following properties:

(1) u has no R-smallest element, and

(2) every nonempty subset of x with fewer elements than u does have an R-smallest element.

Let a be an element of u. By assumption u is not empty. Then $u \sim \{a\}$ is nonempty (why?) and has fewer elements than u. So by property (2), $u \sim \{a\}$ must have an R-smallest element, say s. Then we have,

(3) sRb, for all $b \in u \sim \{a\}$.

Since R is connected on x, we must have that

(4) sRa or aRs.

If sRa holds then it follows from (3) that s is the R-smallest element of u, and this contradicts (1). If, on the other hand, aRs holds then, since R is transitive, it follows from (3) that aRb holds for all $b \in u \sim \{a\}$. This implies that a is the R-smallest element of u, and again we contradict (1). Consequently, the assumption that there is a subset of x with no R-smallest element is false, so R must well-order x.

Now, to prove the converse, we assume x can be linearly ordered, and every linear ordering of x is a well-ordering. Suppose, then, that R linearly orders x. We shall show that R^{-1} also linearly orders x so, by hypothesis, both R and R^{-1} well-order x. Then it follows from Theorem 2 that x is finite. Consequently, to prove the theorem it is sufficient to show that R^{-1} linearly orders x. Recall that

$$aR^{-1}b \text{ iff } bRa.$$

Now, looking back at the definition of a linear ordering, one sees that it is very easy to show R^{-1} is a linear ordering, assuming R is, thus completing the proof.

We just sketch the proof of the next theorem and let the reader fill in the details.

THEOREM 4. *A set x is finite iff $\mathscr{P}(x)$ is the only set S which satisfies the following four conditions*:

(i) $S \subseteq \mathscr{P}(x)$.

(ii) $\varnothing \in S$.

(iii) *If $a \in x$ then $\{a\} \in S$.*

(iv) *If $y, z \in S$ then $y \cup z \in S$.*

Proof. For any set x, $\mathscr{P}(x)$ does satisfy conditions (i)–(iv). We show first that if x is finite and S satisfies conditions (i)–(iv), then $S = \mathscr{P}(x)$. Suppose then that x and S do satisfy these assumptions. Let y be any subset of x. Then y is finite, so there is a natural number n such that

$$y = \{a_1, a_2, \cdots, a_n\}.$$

Then using properties (ii)–(iv) and mathematical induction on n we prove $y \in S$, from which it follows that

(v) $\mathscr{P}(x) \subseteq S$.

Thus, from (i) and (v) we obtain $S = \mathscr{P}(x)$.

Conversely, assume $\mathscr{P}(x)$ is the only set S which satisfies the four conditions. Then show that $E(x)$, the set of all finite subsets of x, satisfies the four conditions. Thus, we must have $E(x) = \mathscr{P}(x)$, which implies that every subset of x is finite. But $x \subseteq x$, so x must be finite.

The proof of our next theorem again illustrates the use of mathematical induction in constructing a function.

THEOREM 5. *A set x is finite iff there is a function F which maps x into itself but no proper subset of x is mapped into itself by F.*

Proof. Suppose x is finite but $x \neq \varnothing$. Then there is a natural number n such that

$$x = \{a_1, a_2, \cdots, a_n\}.$$

Define F on x such that if $i = 1, 2, \cdots, n-1$, then $F(a_i) = a_{i+1}$ and $F(a_n) = a_1$. Then F is the required function. If $x = \varnothing$ we take $F = \varnothing$, and all the properties are satisfied.

Conversely, suppose F is a function satisfying the conditions in the theorem. If $x = \varnothing$, then it is finite and we are through. So suppose $x \neq \varnothing$ and $a \in x$. Define a function G with domain N as follows:

$$G(0) = a$$

$$G(n+1) = F(G(n)) \text{ for all } n \in N.$$

Then we have

$$G(0) = a$$

$$G(1) = F(a)$$

$$G(2) = F(F(a))$$

$$G(3) = F(F(F(a)))$$

etc.

Let y be the range of G. Since F maps x into itself, $y \subseteq x$. But F also maps y into itself, so we must have $y = x$, since F cannot map any proper subset of x into itself. Suppose there is no natural number $n > 0$ such that $G(n) = a$. Then $x \sim \{a\}$ is a proper subset of x which is mapped into itself by F. This is impossible. Let m be the smallest natural number $n > 0$ such that $G(0) = G(n)$; that is, m is the smallest natural number in the set $\{n : n > 0 \text{ and } G(0) = G(n)\}$. Then

$$x \subseteq \{G(0), G(1), \cdots, G(m-1)\}.$$

Consequently, x is a subset of a finite set and must be finite.

It follows from property (V) of finite sets that no finite set is equipollent to a proper subset of itself. In the early part of the twentieth century Richard Dedekind proposed that this property might be used to define a finite set. We call a set with this property *Dedekind finite*. Thus, we give the following definition:

DEFINITION 2. *A set x is Dedekind finite if x is not equipollent to any proper subset of itself.*

It follows from property (V) that every finite set is Dedekind finite. However, it is a consequence of the recent work of Paul Cohen [**1**] that the converse cannot be proved without using an additional axiom of set theory, the axiom of choice. The notion of Dedekind finite being interesting in itself, we give some equivalent formulations of it. Then to conclude the paper we prove that a set x is finite iff $\mathscr{P}(\mathscr{P}(x))$ is Dedekind finite.

A set x is said to be *countably infinite* iff $N \approx x$.

THEOREM 6. *A set x is Dedekind finite iff x does not contain a countably infinite subset.*

Proof. Suppose x does contain a countably infinite subset y. Then $N \approx y$. Let F be a bijection with domain N and range y.

$$F: N \approx y.$$

Let G be a function with domain x defined as follows:

$$G(u) = \begin{cases} u & \text{if } u \in x \sim y \\ F(n+1) \text{ if } u \in y & \text{and} \quad u = F(n). \end{cases}$$

Thus, G is the identity function on $x \sim y$, and G shifts each element of y up to the next element. Consequently, G is a 1-1 function and the range of G is $x \sim \{G(0)\}$ which is a proper subset of x. Therefore, x is equipollent to a proper subset of itself, so x is not Dedekind finite. It follows that if x is Dedekind finite, x does not contain a countably infinite subset.

Conversely, suppose x is not Dedekind finite. Then there is a proper subset y of x and a bijection F such that

$$F : x \approx y.$$

Let $a \in x \sim y$. Using mathematical induction as we did in the proof of Theorem 5 we construct a function G with domain N and such that

$$\begin{aligned} G(0) &= a &&= F^{(0)}(a) \\ G(1) &= F(a) &&= F^{(1)}(a) \\ G(2) &= F(F(a)) &&= F^{(2)}(a) \\ G(3) &= F(F(F(a)) &&= F^{(3)}(a) \\ &\text{etc.} \end{aligned}$$

We claim G is a 1-1 function. For suppose there are natural numbers m and n with $m \leqq n$ such that

$$G(m) = G(n).$$

Then

$$F^{(m)}(a) = F^{(n)}(a).$$

But F is a 1-1 function so if we apply F^{-1}, the inverse of F, to both sides of the preceding equation m times we obtain

$$a = F^{(n-m)}(a).$$

However, $a \in x \sim y$ and if $n - m > 0$, $F^{(n-m)}(a) \in y$, so this last equation can hold only if $n = m$. Hence G is a 1-1 function mapping N into x, so the range of G is a countably infinite subset of x.

In our next theorem, we use the symbol "$\prec$", where $x \prec y$ iff there is a 1-1 function mapping x into y but it is not the case that $x \approx y$.

THEOREM 7. *A set x is Dedekind finite iff $x \prec x^+ = x \cup \{x\}$.*

Proof. Note first that if x is Dedekind finite, so is x^+. Clearly $x \subseteq x^+$, so the identity function is a 1-1 function mapping x into x^+. If $x \approx x^+$ then x^+ would be equipollent to a proper subset of itself, contradicting the fact that x^+ is Dedekind finite. Therefore, $x \prec x^+$. In this argument we have used the fact that $x \notin x$ to ensure that x is a proper subset of $x \cup \{x\}$. However, in the theorem we could replace "x^+" by any set which has one more element than x.

Conversely, if x is not Dedekind finite, then, by Theorem 6, x has a countably infinite subset y. Suppose F is a function such that

$$F: N \approx y.$$

Define a function G with domain x^+ such that

$$\begin{aligned} G(x) &= F(0) & & \\ G(F(n)) &= F(n+1) & & \text{for all } n \in N \\ G(u) &= u & & \text{if } u \in x \sim y. \end{aligned}$$

Then, it is easy to see that

$$G: x^+ \approx x$$

which implies that it is not the case that $x \prec x^+$.

Now here is the promised final result.

THEOREM 8. *A set x is finite iff $\mathscr{P}(\mathscr{P}(x))$ is Dedekind finite.*

Proof. If x is finite, then by property (IV), $\mathscr{P}(\mathscr{P}(x))$ is finite and by property (V) every finite set is Dedekind finite.

Suppose x is not finite. For each $n \in N$, let S_n be the set of all subsets of x which have n elements. Thus,

$$\begin{aligned} S_0 &= \{\varnothing\} \\ S_1 &= \{\{u\}: u \in x\} \\ S_2 &= \{\{u, v\}: u, v \in x \text{ and } u \neq v\} \end{aligned}$$

etc.

It follows from the assumption that x is not finite that $S_n \neq \varnothing$ for all $n \in N$, so for all $n, m \in N$ such that $n \neq m$, we have $S_n \neq S_m$. Moreover, for each $n \in N$, $S_n \in \mathscr{P}(\mathscr{P}(x))$. Therefore, the set

$$\{S_n: n \in N\}$$

is a countably infinite subset of $\mathscr{P}(\mathscr{P}(x))$, so, it follows from Theorem 6, that $\mathscr{P}(\mathscr{P}(x))$ is not Dedekind finite. This completes the proof of the theorem.

In conclusion we summarize the principal results of this paper.

(A) The following statements are equivalent:

(1) x is finite.

(2) x is equipollent to a natural number.

(3) Every nonempty subset of $\mathscr{P}(x)$ has a $\subseteq$-maximal element.
(4) There is a relation R such that both R and R^{-1} well-order x.
(5) x can be linearly ordered, and every linear ordering of x is a well-ordering.
(6) $\mathscr{P}(x)$ is the only set S which satisfies the conditions
(i) $S \subseteq \mathscr{P}(x)$.
(ii) $\varnothing \in S$.
(iii) If $a \in x$ then $\{a\} \in S$.
(iv) If $y, z \in S$ then $y \cup z \in S$.
(7) There is a function F which maps x into itself, but no proper subset of x is mapped into itself by F.
(8) $\mathscr{P}(\mathscr{P}(x))$ is Dedekind finite.

(B) The following statements are equivalent:
(1) x is Dedekind finite.
(2) x is not equipollent to any proper subset of itself.
(3) x does not contain a countably infinite subset.
(4) $x \prec x^+$.

It is interesting to see the wide variety of ways to characterize the notion of finiteness. The reader is encouraged to find additional ways of his own.

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THE COMPUTER AND BASIC STATISTICS: AN EXAMPLE

DONALD E. HENSCHEL and WALTER J. WADYCKI, University of Illinois at Chicago Circle

1. Introduction. A recurrent theme in the proceedings of a recent international conference on the teaching of statistics is that the computer has become an important tool in statistical training (see [**1**]). The "aid to learning" and the "digital viewgraph" potentials of the computer as an instrument for teaching have been mentioned by David Wallace [**2**]. The purpose of this paper is to report on our experience in a basic statistics course with a computer program which illustrates the value of the computer in these two roles. An attractive feature of the approach we have used is that students need no skill in computer programming. The versatility of the program in illustrating both probability and inferential principles is made evident below.

2. CLT computer program. The Central Limit Theorem is one of the most difficult concepts for students studying basic statistics. Although the theorem can be stated in a rather elementary form, most students never really acquire an intuitive feeling for this natural phenomenon upon which much of statistical inference is based. This lack of understanding occurs because students are seldom given an opportunity to see this phenomenon at work. In order to present the Central Limit Theorem to students in a way which involves them in the process, we have written a computer program — CLT — which illustrates the mechanism of the Central Limit Theorem.

CLT illustrates the Central Limit Theorem by exhibiting empirical sampling distributions generated through simulating a large number of random samples, each of the same size, from a Bernoulli process with the probability of success equal to π. As a concrete example of a Bernoulli process we may take the toss of a fair die; if our interest is in observing an ace (the side of the cube which has one dot on it), π would equal one-sixth. We take an empirical approach to determine the distribution of the proportion of aces in a sample of size n. In effect, we use the random number generator of the computer as a giant "bead box." If the uniformly distributed random number is less than one-sixth, we would judge the outcome as a success (an ace) and assign a one as the value of that sample element and zero otherwise. After all n elements of the sample have been selected in this way, the proportion of aces is calculated. This process is repeated a large number of times (say two hundred and fifty) to generate a distribution of proportions which will be close to the theoretical ideal.

The CLT program prints out the sample proportion (P) for each of the samples generated. For the complete simulation, the proportions are summarized both graphically with a histogram and numerically with descriptive statistics such as the mean and standard error. If samples of different size (n) are requested, the histograms corresponding to these samples will tend to look more like a normal curve for large values of n than for small values of n. Thus, by specifying a set of increasing sample sizes, the various sampling distributions of P can be observed as they approach the normal. Also this process can be repeated for several different values of π, thereby affording the students an opportunity to see how the skewness of the population distribution affects the various stages of the sampling distributions of P.

The program is designed to be both versatile and easy to use. The input consists of a few cards on which the students indicate (a) the value of π, (b) the number of samples of size n that are to be generated, (c) a starting value for the random number generator, (d) up to seven different values of n, (e) the upper and lower bounds for each histogram, and (f) the number of equal intervals into which each histogram is to be partitioned (maximum of twenty intervals). A large number of samples may be generated if desired since the program executes quickly. We have run large jobs requiring the generation of 830,000 random numbers and the plotting of fourteen histograms which have taken only about 1.5 minutes of central processor time on an IBM 370/155 computer.

Some of the output from these jobs has been included in this article to illustrate how the program may be used. Due to space limitations only a few of the histograms are shown. We selected nineteen class intervals on the horizontal axis for each diagram. The choice of upper and lower bounds for values on this axis insures that the theoretical mean (π) of each distribution occurs as the midpoint of the tenth class interval.

FIG. 1. CLT Histogram for $n = 1$ and $\pi = 0.10$[a]

Frequency	0	0	0	0	0	0	0	0	0	219	31	0	0	0	0	0	0	0	0

Each * equals 5 points

215										*									
210										*									
205										*									
200										*									
195										*									
190										*									
185										*									
180										*									
175										*									
170										*									
165										*									
160										*									
155										*									
150										*									
145										*									
140										*									
135										*									
130										*									
125										*									
120										*									
115										*									
110										*									
105										*									
100										*									
95										*									
90										*									
85										*									
80										*									
75										*									
70										*									
65										*									
60										*									
55										*									
50										*									
45										*									
40										*									
35										*									
30										*	*								
25										*	*								
20										*	*								
15										*	*								
10										*	*								
5										*	*								
Interval class	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

[a] Extreme bounds of the horizontal axis are -8.4 and 8.6 which yield a unit length for each interval class; class interval ten has limits -0.4 and 0.6. Note that the first interval class and the nineteenth interval class are open-ended; interval one contains all values less than -8.4.

FIG. 2. CLT Histogram for $n = 9$ and $\pi = 0.10$[a]

Frequency	0	0	0	0	0	0	0	0	97	101	36	14	2	0	0	0	0	0	0
Each * equals 3 points																			
99										*									
96									*	*									
93									*	*									
90									*	*									
87									*	*									
84									*	*									
81									*	*									
78									*	*									
75									*	*									
72									*	*									
69									*	*									
66									*	*									
63									*	*									
60									*	*									
57									*	*									
54									*	*									
51									*	*									
48									*	*									
45									*	*									
42									*	*									
39									*	*									
36									*	*	*								
33									*	*	*								
30									*	*	*								
27									*	*	*								
24									*	*	*								
21									*	*	*								
18									*	*	*								
15									*	*	*								
12									*	*	*	*							
9									*	*	*	*							
6									*	*	*	*							
3									*	*	*	*							
Interval class	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

[a] Extreme bounds of horizontal axis selected to yield one-tenth as cell length; class interval ten has endpoints of 0.05 and 0.15.

The set of four figures represents a sequence of samples drawn from a population which is badly skewed (only ten per cent of the population has the characteristic of interest). Each figure is based on 250 samples of size n, where n varies from figure to figure; the values of n used were 1, 9, 25, and 900. The empirical sampling distributions generated by CLT and plotted in the histograms dramatically approach the bell-shaped normal as n increases.

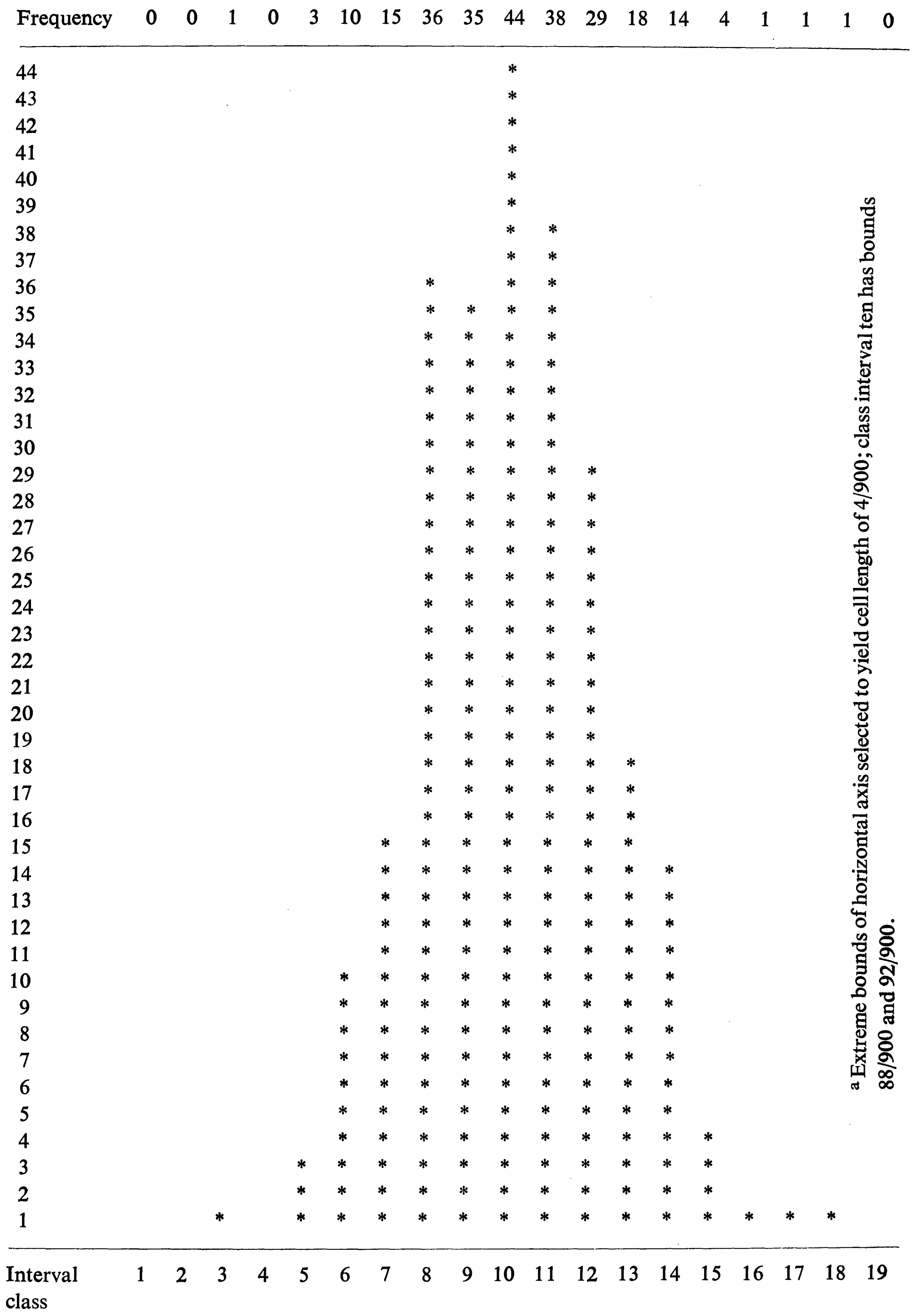
FIG. 3. CLT Histogram for $n = 25$ and $\pi = 0.10$[a]

```
Frequency    0   0   0   0   0   0   0  18  54  65  52  38  13   6   3   1   0   0   0
----------------------------------------------------------------------------------------
Each * equals 2 points
64                                               *
62                                               *
60                                               *
58                                               *
56                                               *
54                                           *   *
52                                           *   *   *
50                                           *   *   *
48                                           *   *   *
46                                           *   *   *
44                                           *   *   *
42                                           *   *   *
40                                           *   *   *
38                                           *   *   *   *
36                                           *   *   *   *
34                                           *   *   *   *
32                                           *   *   *   *
30                                           *   *   *   *
28                                           *   *   *   *
26                                           *   *   *   *
24                                           *   *   *   *
22                                           *   *   *   *
20                                           *   *   *   *
18                                       *   *   *   *   *
16                                       *   *   *   *   *
14                                       *   *   *   *   *
12                                       *   *   *   *   *   *
10                                       *   *   *   *   *   *
 8                                       *   *   *   *   *   *
 6                                       *   *   *   *   *   *   *
 4                                       *   *   *   *   *   *   *
 2                                       *   *   *   *   *   *   *   *
----------------------------------------------------------------------------------------
Interval     1   2   3   4   5   6   7   8   9  10  11  12  13  14  15  16  17  18  19
class
```

[a] Extreme bounds of the horizontal axis are −0.24 and 0.44 to yield a cell length of 0.04; interval class ten has limits 0.08 to 0.12.

CLT also allows the instructor to disguise the value of π. We have used this option as an aid in teaching the principles of statistical inference. The students have CLT generate a sample (or several samples of different sizes) from a Bernoulli process with the value of π known only to the instructor. They then use the sample proportion from CLT along with the statistical theory to find a confidence interval for π. Once everyone has made an interval estimate, the true value of π is revealed and it is determined how many students have an interval containing π .For example, if the students

Fig. 4. CLT Histogram for $n = 900$ and $\pi = 0.10$[a]



[a] Extreme bounds of horizontal axis selected to yield cell length of 4/900; class interval ten has bounds 88/900 and 92/900.

started with different random numbers, about 90% of them should have π in their intervals, if 90% is the confidence level used.

3. Concluding remarks. We cover the Central Limit Theorem early in the second quarter of a three-quarter sequence in basic statistics. In addition to the uses for CLT outlined above, we have found that the program is an excellent vehicle to introduce business students to the computer. This early exposure has paid dividends; in the third quarter, our students are able to use several "canned" computer programs for multivariate analysis with little assistance from the instructor.

The CLT program is written in FORTRAN and consists of a main program with several subroutines that are modifications of those contained in IBM's Scientific Subroutine Package. A listing of the program and some examples of input cards may be obtained by written request to Professor Wadycki.

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UNIQUE PRIME FACTORIZATION AND LATTICE POINTS

CARL DE BOOR and I. J. SCHOENBERG, University of Wisconsin

The usual way to establish the uniqueness of prime factorization is based on Euclid's Lemma (Corollary 3 below) which in turn is based on Euclid's algorithm. However, for students of meager background and interests ("I must get a C in this course or else the School of ··· will not give me credit for it"), the following geometric approach might be of interest. We assume that the students are familiar with the relevant material, including the factorization of integers into prime factors. Our main concern is the question of uniqueness. As we use the relation $P = (a, b)$ to express that a and b are the coordinates of P (see Figure 1), we reserve the notation $d = \langle a, b\rangle$ to mean that d is the greatest common divisor (g.c.d.) of a and b. In particular, $\langle a, b\rangle = 1$ means that a and b are relatively prime.

1. A geometric determination of $d = \langle a, b\rangle$. The tools are pencil, ruler and square graph-paper. The vertices of the squares are called *lattice points*, and the entire set of lattice points is denoted by $\mathbb{L}$. Selecting a perpendicular coordinate system xOy along two lines of the graph paper, we may equivalently define $\mathbb{L}$ as

(1) $$\mathbb{L} = \{(m, n);\ m \text{ and } n \text{ are integers}\}.$$

We shall often use the following property of the set $\mathbb{L}$:

(2) *If* $A \in \mathbb{L}$, $B \in \mathbb{L}$, $C \in \mathbb{L}$ *and* $\overrightarrow{AB} = \overrightarrow{CD}$, *then also* $D \in \mathbb{L}$.

This can be easily established in different ways.

Let $OAPB$ be a rectangle of dimensions $OA = a$, $OB = b$, where a and b are natural numbers (Figure 1). We draw the diagonal OP and denote by $[OP]$ the diagonal considered as a set of points.

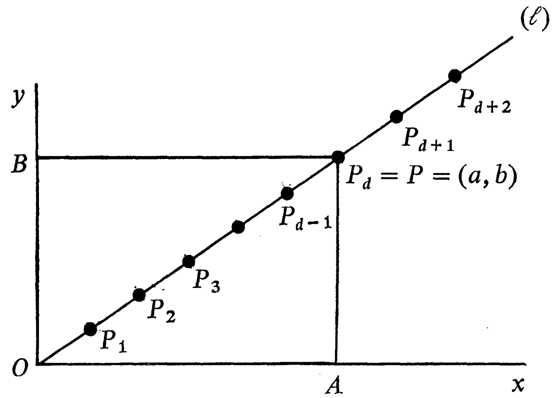

FIG. 1.

Problem 1. *What is the nature of the set*

(3) $$S = [OP] \cap \mathbb{L}?$$

What is the number $|S|$ *of its elements?*

The students are invited to discover the answers to these questions by experimenting with rectangles of various dimensions a and b. The brighter ones may come up with all, or some of the results stated in the following theorem:

THEOREM 1. *The set* S *is composed of a sequence of* $d+1$ $(\geqq 2)$ *equidistant points of which* O *is the first and* P *the last. Hence*

(4) $$S = \{O, P_1, P_2, \cdots, P_{d-1}, P\}$$

and

(5) $$OP_1 = P_1P_2 = \cdots = P_{d-1}P.$$

Moreover

(6) $$d = \langle a, b \rangle.$$

Proof. (i) We extend the diagonal OP indefinitely beyond P obtaining a half-line, or ray, that we denote by l. Let us determine the set

(7) $$R = l \cap \mathbb{L}.$$

Notice that O and P are points of R. Let P', P'', $P^{(3)}, \cdots, P^{(n)}, \cdots$ be points of l such that $OP = PP' = P'P'' = \cdots$. By the property (2) we conclude that all these points $P^{(n)}$ belong to R. Thus R contains infinitely many points. On the other hand, two distinct points U and V of R (and therefore of $\mathbb{L}$) cannot come arbitrarily close together, because $UV \geqq 1$. It follows that the set R can be explicitly represented in the form

(8) $$R = \{O, P_1, P_2, \cdots, P_n, \cdots\}.$$

This means: The points as listed are pairwise distinct and every point of R is among them. They are so numbered that they succeed each other steadily as we proceed along l from O.

We claim the following: *The sequence of points P_n ($P_0 = O$) is equidistant, hence*

(9) $$OP_1 = P_1P_2 = \cdots = P_nP_{n+1} = \cdots.$$

For, if $P_iP_{i+1} < P_jP_{j+1}$, say, let P' be between P_j and P_{j+1} such that $P_iP_{i+1} = P_jP'$. By (2) we conclude that $P' \in R$, while P' is clearly not among the points listed in (8). This contradiction proves the statement. Since $P \in R$, we must have $P = P_d$ for some $d \geqq 1$. This establishes the statements (4) and (5) of Theorem 1.

(ii) We are yet to establish (6). Referring to (9) and Figure 1, let

(10) $$P_1 = (a_1, b_1),$$

and therefore

(11) $$P_j = (ja_1, jb_1), \qquad j = 1, 2, \cdots,$$

and in particular

(12) $$a = da_1, \ b = db_1.$$

We claim that

(13) $$\textit{if } \delta \,|\, a \textit{ and } \delta \,|\, b, \textit{ then } \delta \,|\, d.$$

For if

(14) $$a = \delta a', \ b = \delta b'$$

then $b'/a' = b/a$ and $\delta \geqq 1$ show that $P' = (a', b') \in S$. Now (4) and (11) show that

$$a' = ja_1, \ b' = jb_1, \text{ for some } j,$$

and by (14)

$$a = \delta j a_1, \ b = \delta j b_1.$$

A comparison with (12) shows that $d = \delta j$ and (13) is established. The property (13) clearly establishes (6).

COROLLARY 1. *The numbers a_1 and b_1 are relatively prime, hence*

(15) $$\langle a_1, b_1 \rangle = 1.$$

Proof. Since there are no lattice points on l strictly between O and $P_1 = (a_1, b_1)$, (15) follows from the theorem.

2. A few applications. From (12) and (15) we obtain

$$b/a = b_1/a_1, \ \langle a_1, b_1\rangle = 1. \tag{16}$$

This reduction of the fraction b/a to simplest terms could conceivably be done in different ways. That there is only one way is asserted by

COROLLARY 2. *If also*

$$b/a = b'/a', \ \langle a', b'\rangle = 1, \tag{17}$$

then

$$a' = a_1, \ b' = b_1 \tag{18}$$

and therefore

$$a = da', \ b = db', \ d = \langle a, b\rangle, \tag{19}$$

by (12).

Proof. The first relation (17) shows that $P' = (a', b') \in R$, and from (8) and (11) we conclude that $a' = ja_1$, $b' = jb_1$, for some j. Now the second relation (17) shows that $j = 1$, and (18) is established.

COROLLARY 3 (Euclid's Lemma). *If*

$$a \mid bc, \ \langle a, b\rangle = 1 \tag{20}$$

then

$$a \mid c. \tag{21}$$

Proof. The first relation (20) means that $bc = aq$ for some integer q. Therefore

$$q/c = b/a, \text{ and } \langle a, b\rangle = 1.$$

By Corollary 2 we conclude that $c = a \cdot \langle c, q\rangle$ and (21) is established.

At this point, we may consider the uniqueness of prime factorization as established in the usual way.

3. A theorem of G. Pick. The set $\mathbb{L}$ of lattice points is the discrete version of the plane and its study belongs to the geometry of numbers. See [**2**, Chap. III] for an introduction into this very attractive subject. An interesting result at our level of discussion is as follows (see [**3**] and [**1**, 209]).

THEOREM OF G. PICK. *Let* $\Pi = A_1A_2 \cdots A_n$ *be a simple closed polygon inscribed in* $\mathbb{L}$. *By this we mean that all* $A_i \in \mathbb{L}$. *Let* I *denote the number of lattice points which are in the interior of* Π *and let* B *denote the number of lattice points on the boundary of* Π. *Then*

$$\text{Area of } \Pi = I + (B/2) - 1. \tag{22}$$

The students are asked to draw a prescribed polygon Π and to determine the numbers I and B graphically. They should begin by locating the lattice points on each of the sides of Π by means of our Theorem 1, thereby obtaining the number B. The number I is then easily determined by direct counting, and the right side of (22) is thereby evaluated. On the other hand, the area of Π can also be determined directly from the diagram in terms of the areas of appropriate rectangles and right-angled triangles having legs parallel to the axes. A dramatic moment is reached when these supposedly equal numbers are confronted.

4. Can a triangle inscribed in $\mathbb{L}$ be equilateral? An important application of unique prime factorization is a proof that $\sqrt{p}$ is irrational if p is a prime number (see [**2**], §§4.3, 4.4). In particular, the relation

(23) $\sqrt{3} = b/a$, *where a, b are integers, is impossible.*

Let us use this to establish the well-known

THEOREM 2. *If A, B, and C are points of $\mathbb{L}$, then*

(24) *the triangle ABC cannot be equilateral.*

Proof. We start with a preliminary remark. Let Π be a polygon of the kind appearing in Pick's theorem. That

(25) $$\text{Area of } \Pi = m/2, \quad (m \text{ integer}),$$

is clear from (22). As we have not proved Pick's theorem, we cannot use this argument. However, a direct proof of (25) is immediate as follows: It was pointed out at the end of Section 3 that the area of Π can be obtained by additions and subtractions of the areas of appropriate rectangles, having *integer areas*, and areas of right-angled triangles, which are halves of rectangles, and have therefore *areas of the form $m/2$, where m is integer*. Therefore (25) holds.

Let now ABC be inscribed in $\mathbb{L}$ and let us assume that

(26) *the triangle ABC is equilateral.*

If $a = AB$ is the base of the triangle, then its altitude is $a\sqrt{3}/2$, and therefore

$$\text{Area of } ABC = a^2\sqrt{3}/4.$$

However, a^2 is an integer by the Pythagorean theorem. Using (25), we find that $\sqrt{3}$ equals a rational fraction. This contradicts (23) and establishes the theorem.

The problem extends to higher dimensions as follows. Let $\mathbb{L}_n$ be the set of points of the n-dim. space having integer coordinates in some orthonormal coordinate system. *For what dimensions n can a regular n-dim. simplex be inscribed in* $\mathbb{L}_n$? For a complete answer see [**4**]. The impossibility for $n = 2$ has just been established. However, for $n = 3$ the answer is in the affirmative: A regular tetrahedron can be inscribed in $\mathbb{L}_3$, *even in a single cube*, as first noticed by Kepler. The next lowest dimension with an affirmative answer is $n = 7$.

In concluding let us point out that Theorem 1 and its proof extend from $\mathbb{L} = \mathbb{L}_2$ to the n-dim. case of $\mathbb{Z}_n$. The rectangle $OAPB$ has to be replaced by the rectangular parallelepiped whose diagonal is OP, where $O = (0, \cdots, 0)$, $P = (a_1, a_2, \cdots, a_n)$. The only change in the statement of Theorem 1 for this case is that (6) has to be replaced by

$$d = \langle a_1, a_2, \cdots, a_n \rangle.$$

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PERFECT SQUARE PATTERNS IN THE PASCAL TRIANGLE

ZALMAN USISKIN, University of Chicago

1. Introduction. Many properties of the elements of various rows and diagonals of Pascal's triangle have long been well known, as a bibliography by Schaaf **[4]** attests. But one very elementary result, a beautiful hexagon property, has been discovered recently by Hoggatt and Hansell **[3]**. This property is as follows: Surrounding each element $\binom{n}{r}$ in the usual triangular array of Pascal's triangle are six elements which occupy vertices of a regular hexagon.

$$\binom{n-1}{r-1}, \quad \binom{n}{r+1}, \quad \binom{n+1}{r}, \quad \binom{n-1}{r}, \quad \binom{n}{r-1}, \quad \binom{n+1}{r+1}$$

The product of these six binomial coefficients is a perfect square; moreover, the product of the first three equals the product of the last three. (I am indebted to G. L. Alexanderson for bringing this theorem to my attention.)

Representing the first three coefficients above by O's and the last three by X's, a nice pattern (a) can be pictured. (Think of the dots as being other elements in the triangular lattice of binomial coefficients.) A specific case is given at right.

(a) · · O X · 7 21

· · X · O · 8 $\binom{8}{2}$ 56

· · O X · 36 84

$$(7)(56)(36) = (8)(21)(84)$$

This result was immediately generalized by Hoggatt and Alexanderson **[2]** to multinomial coefficients. Hoggatt and Hansell [3] suggested the existence of extensions to generalized binomial coefficients and ways of building up serpentine or snowflake patterns of equal products. Gould [1] found specific equal products involving 8 or 10 elements. Two of Gould's patterns of equal products are (b) and (c).

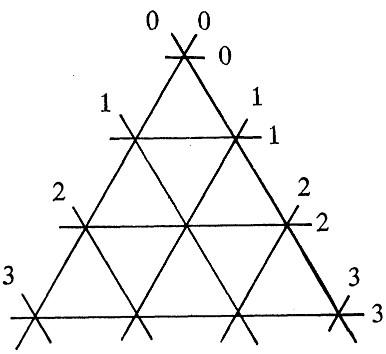
```
(b)     O   X   ·
      ·   ·   O   X
    X   ·   ·   ·   O
      O   X   ·   ·
        ·   O   X
```

```
(c)      ·   O   X   ·
       X   ·   ·   O
       O   ·   ·   X
         ·   X   O   ·
```

When such a pattern is independent of the choice of n and r, it shall be called a perfect square pattern or PSP, for the product of all elements in the pattern is a perfect square.

This note has three purposes: (1) to display a simple sufficient condition for PSP's; (2) to present some interesting PSP's not in the earlier literature; and (3) to suggest possible avenues of further exploration.

2. Sufficient condition.



The major "lines" in the Pascal triangle are horizontal or "major diagonals," slanted at 120° and 240° to the horizontal. These lines can be numbered from the top and from the sides, as the above drawing shows. When this is done, the element $\binom{n}{r}$ lies on the nth horizontal line and the rth and $(n-r)$th major diagonals. These numbers are exactly the factorials which are involved in $\binom{n}{r}$.

In the normal $n!/[(n-r)!\,r!]$ notation for $\binom{n}{r}$, the numerator is determined by the horizontal row, the denominator by the two major diagonals containing $\binom{n}{r}$. To show the equality of products of these coefficients, one needs only to have the same factorials in the numerators and denominators of the two groups of coefficients. This gives a sufficient condition:

THEOREM. *If a pattern contains an even number of elements on every horizontal line and major diagonal, then the pattern is a* PSP.

To illustrate this theorem, we give the following pattern which seems to be new.

(d)

```
  O   .   .   .   X
X   X   .   .   O   O
  O   .   .   .   X
```

If the left element is $\binom{n}{r}$, this pattern can be easily verified algebraically (the drawing pictures the case $a = 4$).

$$\binom{n-1}{r+a}\cdot\binom{n}{r}\cdot\binom{n}{r+1}\cdot\binom{n+1}{r+a+1}=\binom{n-1}{r}\cdot\binom{n}{r+a}\cdot\binom{n}{r+a+1}\cdot\binom{n+1}{r+1}.$$

From any given PSP, the theorem implies that certain transformations generate other PSP's. Among these transformations are translations in any direction, reflections over any major diagonal or the horizontal, rotations of 120° and 240°, and dilatations with any point $\binom{n}{r}$ as center.

3. New perfect square patterns. Example (e) is constructed from the original hexagon (a) using an idea noted by Hoggatt and Hansell—that if two hexagons have an element in common, the common element can be deleted and a PSP arises. This always adds 4 elements to a pattern. Together with pattern (c), this method of enlarging a PSP shows that there exists a PSP with any given even number of elements.

(e)

```
  O   X   O   X
X   .   .   .   O
  O   X   O   X
```

(f)

```
  .   .   .   O   X   .
    .   .   .   .   .
  O   X   .   .   .   .
X   .   .   .   .   .   O
  O   X   O   .   .   X
```

Pattern (f) shows that a PSP need not be either reflection-symmetric or rotation-symmetric. Patterns (g), (h), and (i) provide three examples of aesthetically pleasing PSP's.

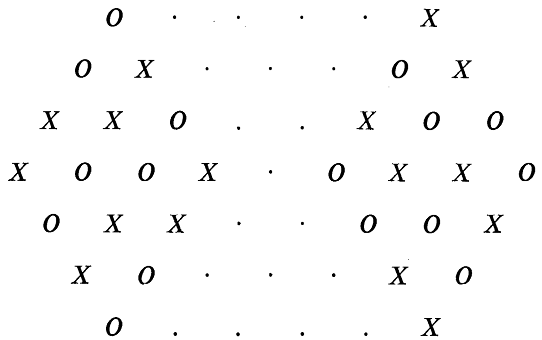
(g)

```
      O   X
    X   .   O
  O   .   .   X
X   .   .   .   O
  O   .   .   X
    X   .   O
      O   X
```

(h)

```
  O   X
X   .   O
  O   .   X   O   X
    X   .   .   .   O
  O   .   X   O   X
X   .   O
  O   X
```

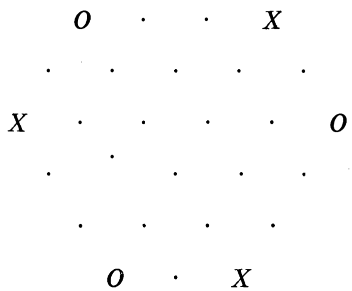
(i)



It is easy to prove that the union of two disjoint PSP's is also a PSP. In fact, (g) is the union of two PSP's while (i) is the union of four PSP's. A great variety of PSP's can be obtained in this way.

PSP (j) is a stretch variant of the original hexagon (a). The corresponding combinatorial identity is next to the figure. Here $k = 2$ and $m = 3$; in general, k and m stand for the lengths of sides of the hexagon. If $k = m$, then the hexagon is regular and so a regular hexagon with any integral side length can be constructed so that the vertices form a PSP.

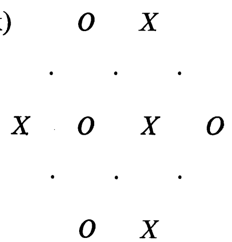
(j)



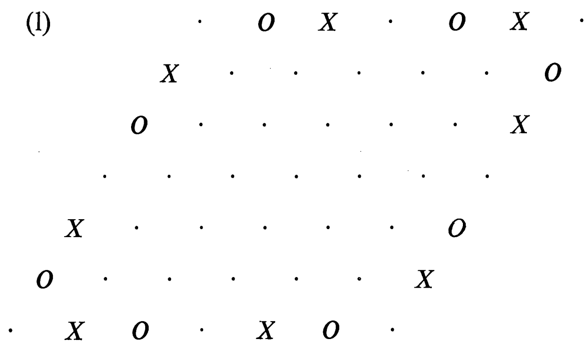
$$\binom{n-k}{r}\cdot\binom{n+m}{r+k}\cdot\binom{n}{r-m} = \binom{n-k}{r-m}\cdot\binom{n+m}{r}\cdot\binom{n}{r+k}$$

PSP (k) is an interesting stretch variant of (d).PSP (1) is the union of two stretch variants. PSP (m) is also the union of two stretched PSP's; adding the hexagon in its interior forms a PSP with 24 elements and only 4 interior points.

(k)



(l)



(m)

```
        O   X
      X   ·   O
   X   O   ·   ·   X   O
 O   ·   ·   ·   ·   ·   X
   X   O   ·   ·   X   O
      X   ·   O
        O   X
```

4. Perfect *n*th power patterns. Figure (n) depicts a perfect *cube* pattern; the product of the X's is equal to the product of the O's, which is in turn equal to the product of the $\#$'s.

(n)

```
      #   ·   ·   O   ·   ·   X
   O   X   ·   X   #   ·   #   O
 X   #   O   #   O   X   O   X   #
   O   X   ·   X   #   ·   #   O
      #   ·   ·   O   ·   ·   X
```

That is, algebraically:

$$\binom{n}{r}\binom{n-2}{r-1}\binom{n+2}{r+1}\binom{n}{r+2}\binom{n}{r-2}\binom{n-1}{r-4}\binom{n-1}{r+3}\binom{n+1}{r-3}\binom{n+1}{r+4}$$

$$=\binom{n-2}{r-4}\binom{n-1}{r}\binom{n-1}{r+2}\binom{n}{r-3}\binom{n}{r-1}\binom{n}{r+4}\binom{n+1}{r+1}\binom{n+1}{r+3}\binom{n+2}{r-2}$$

$$=\binom{n-2}{r+2}\binom{n-1}{r-3}\binom{n-1}{r-1}\binom{n}{r-4}\binom{n}{r+1}\binom{n}{r+3}\binom{n+1}{r-2}\binom{n+1}{r}\binom{n+2}{r+4}.$$

Pattern (n) is easily generalized to yield a perfect nth power pattern by using n $n \times n$ diamonds with n elements cyclically arranged.

5. Possible explorations. (1) Is the sufficient condition given in the theorem also a necessary condition for the existence of a PSP? (It seems so.)

(2) Do the perfect nth power patterns hold for the standard generalizations of the Pascal triangle?

(3) If a PSP is rotation-symmetric about $\binom{n}{r}$, it is easy to prove that $\binom{n}{r}^k$ divides the product of the $2k$ elements in the pattern. Is there a more specific relationship between $\binom{n}{r}$ and the product?

(4) Certain PSP's are vertices of polygons with all sides of length 1. PSP's (a), (e), (g), (h), and (m) are outlines of polygons which have 1, 3, 9, 7, and 11 interior points, respectively. It is easy to show that there exists a PSP polygon with any odd

number of interior points. PSP (o) has 6 interior points. But there does not exist a PSP polygon with 2 interior points. For what n does there exist a PSP polygon with $2n$ interior points? Does there exist any relationship between the number of points in a PSP polygon and the possible number of interior points?

(o)

```
      O   X
    X   ·   O
  O   ·   ·   X
X   ·   ·   ·   O
  O   X   O   X
```

(5) We might call a PSP "indivisible" if it is not the union of two others. Indivisible PSP's might be classified by the numbers of elements along lines and major diagonals, or they might be classified by the number of elements and the symmetries. Is there a convenient classification?

Obviously, the directions are many and perhaps only of interest as a recreation. But there are combinatorial identities associated with every PSP and it is possible that one might use PSP's to generate new identities.

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DUALITY IN SPHERICAL TRIANGLES

MURRAY S. KLAMKIN, Ford Motor Company, Dearborn, Michigan

I. Introduction. In addition to the basic triangle inequalities

$$b + c \geqq a, \qquad c + a \geqq b, \qquad a + b \geqq c,$$

for any spherical triangle, we also usually impose the convexity constraints $a, b, c \leqq \pi$ and $a + b + c = 2s < 2\pi$. By considering the segments of the sides formed by the points of contact of the inscribed circle, as in the plane case [**1**], we are led to a duality between the sides a, b, c and three nonnegative numbers x, y, z denoting the lengths of the segments formed. Here

(1) $$a = y + z, \qquad b = z + x, \qquad c = x + y,$$

and, conversely,

$$x = s - a, \qquad y = s - b, \qquad z = s - c. \tag{2}$$

Also, $x + y + z < \pi$. Consequently, for any valid spherical triangle inequality $F(a, b, c) \geqq 0$, we have the corresponding dual x, y, z inequality $F(y + z,\ z + x,\ x + y) \geqq 0$; for any x, y, z $(x + y + z < \pi)$ inequality $G(x, y, z) \geqq 0$, we have the dual spherical triangle inequality $G(s - a,\ s - b,\ s - c) \geqq 0$. As applications, we derive a number of spherical triangle inequalities which are believed to be new.

II. Applications. One of the most elegant of plane triangle inequalities is the one relating the radii of the circumscribed and inscribed circles, i.e., $R \geqq 2r$, $\{E\}$. ($\{E\}$ denotes with equality iff the triangle is equilateral.) A natural extension for spherical triangles is

$$\tan R \geqq 2 \tan r, \quad \{E\}. \tag{3}$$

The equality case is a known result.

To prove (3), we start off with the following known expressions for R and r:

$$\tan R = \frac{2}{n} \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}, \tag{4}$$

$$\tan r = \frac{n}{\sin s}, \tag{5}$$

where n, the norm of the triangle, is given by

$$n^2 = \sin s \sin (s - a) \sin (s - b) \sin (s - c). \tag{6}$$

Since n is nonnegative, we have to show that

$$\sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2} \geqq \sin (s - a) \sin (s - b) \sin (s - c) \tag{3)'$$

or, in terms of x, y, z, that

$$\sin \frac{y + z}{2} \sin \frac{z + x}{2} \sin \frac{x + y}{2} \geqq \sin x \sin y \sin z. \tag{3)''$$

Since $D^2 \log \sin x = -\csc^2 x$, $\log \sin x$ (as is well known) is concave in $(0, \pi)$. Thus

$$2 \log \sin \frac{y + z}{2} \geqq \log \sin y + \log \sin z.$$

More elementarily,

$$1 \geqq \cos (y - z) = \cos (y + z) + 2 \sin y \sin z,$$

or

$$\sin^2 \frac{y + z}{2} \geqq \sin y \sin z, \tag{7}$$

whence

$$\Pi \sin^2 \frac{y+z}{2} \geqq \Pi \sin y \sin z,$$

which gives (3)″ on taking square roots. Also, since

$$\tan^2 r = \{\Pi \sin x\} / \sin s,$$

$$\tan R \tan r = 2\left\{\Pi \sin \frac{a}{2}\right\} / \sin s,$$

we obtain similarly that

$$\tan^2 r \leqq \frac{\sin^3 s/3}{\sin s}, \qquad \{E\}, \tag{8}$$

$$\tan R \tan r \leqq \frac{2 \sin^3 s/3}{\sin s}, \qquad \{E\}. \tag{9}$$

If in (3), (8), and (9), we let the radius of the sphere increase without limit, we recapture, respectively, the known plane triangle inequalities $R \geqq 2r$, $s^2 \geqq 27r^2$, $2s^2 \geqq 27Rr$.

L. Fejes Tóth in his book, *Regular Figures* [**2**], notes (at least at the time of publication) that there is no direct elementary proof of the isoperimetric property of regular spherical polygons. We give one here for the case of triangles. We shall show that for all spherical triangles on a given sphere with given perimeter, the equilateral one has maximum area. L'Huilier's formula for the spherical excess is

$$\tan^2 \frac{E}{4} = \tan \frac{s}{2} \tan \frac{s-a}{2} \tan \frac{s-b}{2} \tan \frac{s-c}{2}, \tag{10}$$

so that we want to determine

$$\max_{x+y+z=s} \left\{\tan \frac{x}{2} \tan \frac{y}{2} \tan \frac{z}{2}\right\}.$$

Although $\log \tan \theta$ is concave in $(0, \pi/4)$, it is not in $(0, \pi/2)$. However, it is "partially concave" in $(0, \pi/2)$ and this will suffice to solve our problem. The notion of "partial concavity", together with a further duality in spherical triangles is treated more completely in [**3**]. We first show that

$$\tan^2 \frac{u+v}{2} \geqq \tan u \ \tan v, \tag{11}$$

where $u, v \geqq 0$, $u + v \leqq \pi/2$ and with equality iff $u = v$. Rewriting (11) as

$$\frac{1-\cos(u+v)}{1+\cos(u+v)} \geqq \frac{\sin u}{\cos u} \frac{\sin v}{\cos v}$$

and expanding out, gives the obvious inequality

$$\{1 - \cos(u - v)\} \cos(u + v) \geqq 0.$$

From (11), we then have

$$\tan u \tan v \tan w \tan \frac{u+v+w}{3} \leqq \tan^2 \frac{u+v}{2} \tan^2 \frac{u+v+4w}{6} \leqq \tan^4 \frac{u+v+w}{3}, \tag{12}$$

or

$$\tan^3 \frac{u+v+w}{3} \geqq \tan u \tan v \tan w. \tag{13}$$

For (12) to be valid, $u, v, w \geqq 0$, $u + v < \pi/2$, $u + v + 4w < 3\pi/2$, $u + v + w < 3\pi/4$. Consequently, (13) is valid if $u + v + w < \pi/2$ with equality iff $u = v = w$ (note that in (12) we choose w to be the smallest of the three angles). Finally, using (13), E in (10) is a maximum for $a = b = c$.

Two other known area formulas are

$$\frac{\sin^2 E/4}{\sin s/2} = \Pi \frac{\sin (s-a)/2}{\cos a/2},$$

$$\frac{\cos^2 E/4}{\cos s/2} = \Pi \frac{\cos (s-a)/2}{\cos a/2}.$$

By virtue of the previous isoperimetric result, we also have

$$\frac{\sin^3 s/6}{\cos^3 s/3} \geqq \Pi \frac{\sin x/2}{\cos (y+z)/2}, \tag{14}$$

$$\frac{\cos^3 s/6}{\cos^3 s/3} \leqq \Pi \frac{\cos x/2}{\cos (y+z)/2}. \tag{15}$$

(Extracted from a preprint of the author on triangle inequalities **[1]**.)

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AN APPARENTLY ALGEBRAIC PROPERTY OF THE INTEGERS

W. P. BERLINGHOFF, Southern Connecticut State College, New Haven

Abstract algebra is often described as the study of operations on sets. Moreover, isomorphic algebraic systems are usually considered algebraically "the same," since an isomorphism must preserve operations, and hence will also preserve any properties which depend solely on the operations. The purpose of this note is to exhibit a property of the integers which appears to be definable solely in terms of addition and multiplication but which is not preserved under isomorphism. Such an example is often useful in a first course in abstract algebra to emphasize both the power and the limitations of a purely formal treatment of operational structures.

Let $(Z, +, \cdot)$ denote the integers with the usual addition and multiplication, the prototypical integral domain. We know that multiplication can be viewed as repeated addition; formally, for any positive integer p and any integer n

$$(*) \qquad p \cdot n = n + n + \cdots + n,$$

where there are p summands on the right side of the equation. Multiplication of two negative integers using (*) is even possible, since the fact that the product of two integers equals the product of their negatives is derivable directly from the domain axioms. Thus, (*) seems to be a strictly operational property of $(Z, +, \cdot)$.

Now let us impose a new additive structure on Z by choosing a fixed nonzero integer k and defining

$$m \oplus n = m + n + k,$$

for any $m, n \in Z$. It is easy to check that $(Z, \oplus)$ is an Abelian group, with identity $-k$ and the inverse of any n given by $-n - 2k$. Moreover, the mapping f from $(Z, \oplus)$ to $(Z, +)$ defined by

$$f(n) = n + k$$

is clearly a group isomorphism. We can now define a well-behaved multiplicative structure on $(Z, \oplus)$ by "carrying the numbers over" to the original integral domain by f, multiplying them there, and then "bringing back" the product by f^{-1}. More precisely, we observe that

$$f^{-1}(f(m) \cdot f(n)) = [(m + k) \cdot (n + k)] - k,$$

and hence define $\otimes$ on Z by

$$m \otimes n = [(m + k) \cdot (n + k)] - k,$$

for all $m, n \in Z$. It is easy to see that $\otimes$ is by definition precisely what is needed, not only to make $(Z, \oplus, \otimes)$ an integral domain, but also to guarantee that f is a ring isomorphism from $(Z, \oplus, \otimes)$ to $(Z, +, \cdot)$.

Consider now the formal analogue in $(Z, \oplus, \otimes)$ of the property (*): For any positive integer p and any integer n,

$$p \otimes n = n \oplus n \oplus \cdots \oplus n,$$

where there are p summands on the right side of the equation. For example, using 2 and 3 we get

$$2 \otimes 3 = [(2 + k) \cdot (3 + k)] - k = 6 + 4k + k^2, \text{ and}$$

$$3 \oplus 3 = 3 + 3 + k = 6 + k,$$

which are obviously unequal since k is nonzero. In fact, since the k in the above construction of $(Z, \oplus, \otimes)$ was arbitrary, we actually have infinitely many integral domain structures on Z, all isomorphic to $(Z, +, \cdot)$, but "multiplication" by p is "addition" of p summands *only* in $(Z, +, \cdot)$ itself!

A moment's reflection reveals that the positive integer p in (*) is actually being used to "count" summands; that is, (*) relates the interpretation of p as positive integer to that of p as cardinal number. Hence, the compatibility of these two roles of p in the integral domain requires that the operations reflect the set-theoretic origins of the positive integers; i.e., that the addition and multiplication be precisely those derived from the corresponding operations of disjoint union and Cartesian product on sets.

It is possible, however, to treat $\otimes$ as repeated $\oplus$ provided the number of summands is chosen differently. In fact, if $(Z, \oplus, \otimes)$ is an arbitrary integral domain structure on the integers and f is an isomorphism from $(Z, \oplus, \otimes)$ to $(Z, +, \cdot)$, then for any integer p such that $f(p)$ is positive and any integer n,

$$p \otimes n = n \oplus n \oplus \cdots \oplus n,$$

where there are $f(p)$ summands on the right side of the equation, since

$$\begin{aligned} p \otimes n &= f^{-1}[f(p) \cdot f(n)] \\ &= f^{-1}[f(n) + f(n) + \cdots + f(n)] \qquad [f(p) \text{ summands}] \\ &= n \oplus n \oplus \cdots \oplus n \qquad [f(p) \text{ summands}]. \end{aligned}$$

Acknowledgement: The author would like to thank Professor James Reid of Wesleyan University for several valuable suggestions, especially the observation in the last paragraph.

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ON NUMBER THEORETIC FUNCTIONS WHICH SATISFY $f(x + y) = f(f(x) + f(y))$

JOSEPH N. SIMONE, University of Missouri, Kansas City

A common equivalence relation on the integers frequently encountered in number theory is $c \equiv a \pmod b$. If $c \equiv a \pmod b$ and $0 \leqq a < b$, then a is said to be the least residue of c modulo b [1], and we write LRes(c, b) for a. Thus, LRes(c, b) is the least nonnegative remainder when c is divided by b and LRes is an arithmetic function for each nonzero b. We develop a functional equation which characterizes the least residue function modulo b when the domain is restricted to the set of nonnegative integers and $b > 0$.

Unless otherwise stated, all small Roman letters are nonnegative integers.

DEFINITION 1. *Let f be a function with domain D, the set of nonnegative integers, and with range $R = \{0, \cdots, m\}$, where m is a fixed nonnegative integer. Then f is said to be a semi-distributive function if and only if:*

(a) $f(a + b) = f(f(a) + f(b))$ *for all* $a, b \in D$;

(b) $f(0) = 0$.

Certain results follow directly from the above definition and are summarized in the following theorem.

THEOREM 1. *Let f, D, R and m be as in Definition 1 and let $a \in D$. Then,*

(a) $f(a) = f(f(a))$;

(b) $f(c) = c$ *for all* $c \in \{0, \cdots, m\}$.

Proof. (a) $f(a) = f(a + 0) = f(f(a) + f(0)) = f(f(a) + 0) = f(f(a))$.

(b) If $c \in \{0, \cdots, m\} = R$, then there exists $b \in D$ such that $f(b) = c$. Hence, $c = f(b) = f(f(b)) = f(c)$.

THEOREM 2. *Let f be a semidistributive function with range $\{0, \cdots, m\}$. Set $a = f(m + 1)$ and $b = m + 1 - a$. Then for all $c \geqq a$ it follows that $f(c + b) = f(c)$.*

Proof. Let $c \geqq a$. Then, $f(c + b) = f(c + m + 1 - a) = f((c - a) + (m + 1)) = f(f(c - a) + f(m + 1)) = f(f(c - a) + f(f(m + 1))) = f(c - a + f(m + 1)) = f(c - a + a) = f(c)$, and this completes the proof.

Theorem 2 states that every semidistributive function is ultimately periodic. We call $a = f(m + 1)$ the *initiator* of f and $b = m + 1 - a$ the *ultimate period* of f. It should be noted that if r is a nonnegative integer and $c \geqq a$, then $f(c + rb) = f(c)$, as may be seen by a simple induction argument.

Our final theorem gives a complete characterization of those functions described by Definition 1.

THEOREM 3. *Let f be a function from $\{0, 1, 2, \cdots\}$ onto $\{0, \cdots, m\}$. Then f is a semidistributive function if and only if there exist $a \geqq 0$ and $b > 0$ such that for all $x \geqq a$, $f(x) = \text{LRes}((x - a), b) + a$, while for all $x < a$, $f(x) = x$.*

Proof. It is easily verified that the above function is a semidistributive function, and this part of the proof is left to the reader.

So, let f be a semidistributive function with initiator a and ultimate period b. Obviously $a \geqq 0$ and $b > 0$. If $x < a$, then $x \in \{0, \cdots, m\}$ and by Theorem 1, $f(x) = x$. Thus, let $x \geqq a$ and $y \geqq 0$. By the division algorithm there exist unique integers r and t such that $0 \leqq t < b$ and $y = rb + t$. From Theorem 2, $b = m + 1 - a$ so $0 \leqq t < m + 1 - a$ or $a \leqq t + a < m + 1$. Thus, $t + a \in \{0, \cdots, m\}$ so $f(y + a) = f(rb + t + a) = f(t + a) = t + a$. However, $t = \text{LRes}(y, b)$, so $f(y + a) = \text{LRes}(y, b) + a$. Hence, if $x \geqq a$, then $x - a \geqq 0$ so $f(x) = f((x - a) + a) = \text{LRes}((x - a), b) + a$, completing the proof.

It is worth noting that the semidistributive functions with positive initiators characterize the finite cyclic semigroups [2]. To see that this is true, suppose that $\{a, a^2, a^3, \cdots\}$ is a cyclic semigroup and there exists a semidistributive function f with positive initiator such that $a^n = a^{f(n)}$ for all positive n. If $\{0, \cdots, m\}$ is the range of f, then $\{a, a^2, \cdots\} = \{a, a^2, \cdots, a^m\}$ and is therefore a finite cyclic semigroup. On the other hand, suppose that $\{a, a^2, \cdots, a^m\}$ is a finite cyclic semigroup. Then for each positive integer n there exists a unique positive integer $g(n)$ such that $1 \leqq g(n) \leqq m$ and $a^n = a^{g(n)}$. Define the function f on the nonnegative integers by letting $f(0) = 0$ and $f(n) = g(n)$ for all $n \geqq 1$. It is easily seen that f is a semidistributive function with positive initiator. If $\{e, a, a^2, \cdots, a^m\}$ is a finite cyclic semigroup with identity, then for each *nonnegative* integer n there exists a unique nonnegative integer $h(n)$ such that $0 \leqq h(n) \leqq m$ and $a^n = a^{h(n)}$. The function h is seen to be a semidistributive function.

Finally, we note that the semidistributive functions with zero initiator characterize the finite cyclic groups, which may be seen by an argument along the same lines as the argument in the preceding paragraph.

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NUMBER OF SOLUTIONS OF THE CONGRUENCE $x^m = r$ (mod n)

JAMES ALONSO, Bennett College, Greensboro, North Carolina

Introduction. The present note makes use of a theorem from *Group Theory* to obtain a result in Number Theory. The reader is familiar with Euler's theorem that states that $x^{\phi(n)} = 1$ (mod n) for every x relatively prime to n. We will count the number of solutions of the more general congruence equation $x^m = r$ (mod n) and Euler's theorem will follow as a consequence.

Throughout this note n, m and r will be fixed natural numbers, $n = \prod_{i=1}^{h} p_i^{b_i}$, $p_1, \cdots, p_h$ distinct primes and $b_i > 0$ for $i = 1, \cdots, h$, $s_i = (m, p_i^{b_i - 1}(p_i - 1))$ and $s = \prod_{i=1}^{h} s_i$.

The following theorem from *Group Theory* will be used:

THEOREM. *If A is a cyclic group of order n, the group* Aut(A) *of the automorphisms of A is isomorphic to the direct sum $\sum_{i=1}^{h} A_i$ where*:

(i) *A_i is a cyclic group of order $p_i^{b_i - 1}(p_i - 1)$ if $p_i > 2$,*

(ii) *A_i is a cyclic group of order $2^{b_i - 1}$ if $p_i = 2$ and $b_i = 1$ or 2, and*

(iii) *$A_i \cong B_1 + B_2$ with B_1 of order 2 and B_2 cyclic of order $2^{b_i - 2}$ if $p_i = 2$ and $b_i \geqq 3$.*

For a proof of this theorem see [**1**] pages 120–121.

THEOREM 1. *If the prime numbers* $p_1, \cdots, p_h$ *are odd, the congruence*

$$x^m = 1 \pmod{n} \tag{1}$$

has exactly s different solutions (mod n).

Proof. Let $A = \langle a \rangle$ be a cyclic group of order n. By the above-mentioned theorem

$$\operatorname{Aut}(A) \cong \sum_{i=1}^{h} A_i \tag{2}$$

where A_i is a cyclic group of order $p_i^{b_i-1}(p_i - 1)$.

(i) The group A_i has exactly s_i elements whose orders divide s_i. If we choose $x_i \in A_i$ $(i = 1, \cdots, h)$ such that $o(x_i) \mid s_i$ then $o(x_1, \cdots, x_h) \mid m$. In this way we obtain s elements of $\sum_{i=1}^{h} A_i$ whose orders divide m. Conversely if $o(x_1, \cdots, x_h) \mid m$ then $o(x_i) \mid m$ and $o(x_i) \mid p_i^{b_i-1}(p_i - 1)$, therefore $o(x_i) \mid s_i$.

This shows that $\sum_{i=1}^{h} A_i$ has exactly s elements whose orders divide m, and correspondingly there are exactly s automorphisms of A whose orders divide m; if t is one of them and $t: a \mapsto a^k$ then $t^m: a \mapsto a^{k^m} = a$ therefore $k^m = 1 \pmod{n}$, which shows that the congruence (1) has at least s solutions.

(ii) Conversely, if k is a solution of (1), i.e., if $k^m = 1 \pmod{n}$ and t is the endomorphism of A determined by $t: a \mapsto a^k$, then $(k, n) = 1$ otherwise $k = p_i z$ for some natural number z and $1 \leqq i \leqq h$, hence $p_i^m z^m = 1 \pmod{n}$ which is impossible. Thus t is automorphism, and since $t^m: a \mapsto a^{k^m} = a$, the order of t divides m; therefore the solution k is one of those found in part (i). This shows that (1) has at most s solutions.

THEOREM 2. *The same statement as in Theorem* 1 *is true if one of the prime numbers, say* p_1, *is* 2 *and its exponent* b_1 *is* 1 *or* 2.

The proof is the same as for Theorem 1.

THEOREM 3. *If one of the prime numbers, say* p_1, *is* 2 *and its exponent* $b_1 \geqq 3$, *then the number of solutions* (mod n) *of* (1) *is*:

(i) s, *if* $s_1 = 1$ (*hence m is odd*) *or* $s_1 = 2^{b_1-1}$,

(ii) $2s$, *if* $1 < s_1 < 2^{b_1-1}$.

The proof is left to the reader and it is similar to the proof of Theorem 1 taking into account that in this case the group A_1 of (2) is isomorphic to $B_1 + B_2$ with B_1 of order 2 and B_2 cyclic of order 2^{b_1-2}.

COROLLARY 1. *If m is a multiple of* $\phi(n)$, *then* (1) *holds for every x relatively prime to n* (Euler's Theorem).

COROLLARY 2. *If the numbers* $p_i^{b_i-1}(p_i - 1)$ $(i = 1, \cdots, h)$ *are relatively prime in pairs, or equivalently, if the group* Aut(A) *of* (2) *is cyclic, then the number of different solutions of* (1) (mod n) *is* $(m, \phi(n))$.

In the case of Corollary 2 there exists $t \in \text{Aut}(A)$ of order $\phi(n)$. If $t: a \mapsto a^k$ then

$$k^{\phi(n)} = 1 \text{ and } k^m \neq 1 \text{ for } 0 < m < \phi(n) \qquad (\text{mod } n).$$

k is said to be a primitive root of n, and $\{k, k^2, \cdots, k^{\phi(n)}\}$ is a complete system of numbers relatively prime to n and incongruent to each other (mod n).

Observe that the conditions of Corollary 2 are satisfied if and only if n is 2 or 4 or p^b or $2p^b$ with p odd prime and $b > 0$.

THEOREM 4. *If* $(r, n) = 1$, *the congruence*

$$x^m = r \ (\text{mod } n) \tag{3}$$

has no solutions or as many solutions (mod n) *as the congruence* (1).

Proof. (i) Assume that $x_1, \cdots, x_z$ $(z \geqq 1)$ are all the different solutions (mod n) of (3); let $y_1, \cdots, y_z$ be numbers such that $x_i y_i = 1$ (mod n). Then $x_1 y_1, \cdots, x_1 y_z$ are different solutions of (1). Therefore (3) has at most as many solutions as (1).

(ii) Conversely, if (3) has one solution, say x_1 and $k_1, \cdots, k_z$ are the different solutions of (1), then $x_1 k_1, \cdots, x_1 k_z$ are distinct solutions of (3), therefore (3) has at least as many solutions as (1).

If we call G the group (isomorphic to Aut(A) of (2)) of the natural numbers less than n and relatively prime to n under multiplication (mod n), we observe that the solutions of (1) form a subgroup H of G, and the solutions of (3) a coset of H. On the other hand the subset R of G of the numbers r such that (3) has a solution, is also a subgroup of G. Since R has exactly one element for each coset of H, we have $[G: H] = |R|$.

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AN ISOPERIMETRIC PROBLEM ON A LATTICE

D. E. DAYKIN, University of Reading, England

1. Introduction. The familiar isoperimetric problem is to show that a circle is that plane closed curve of given length containing maximum area, and here we consider the corresponding problem on a lattice. The problem was posed for a planar lattice in conversation by D. A. Klarner.

Let L be either (i) the set of squares in a plane lattice, or (ii) the set of cubes in a three dimensional lattice. If S is a finite subset of L the boundary $B(S)$ of S is the set of members of L which are not in S but have a side in common with a member of S. Our main result is that if S has minimal $|B(S)|$ then S is roughly diamond-shaped in case (i) and roughly octahedron-shaped in case (ii).

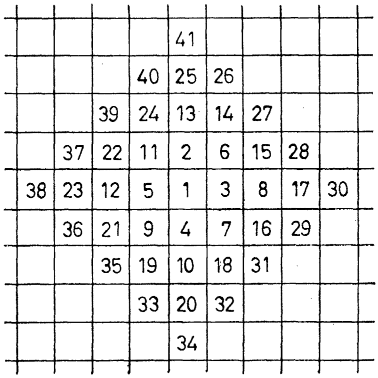

FIG. 1

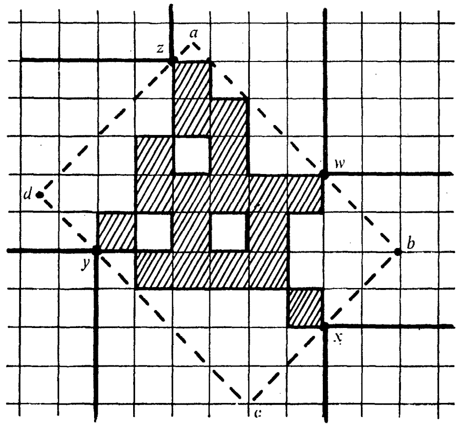

FIG. 2

2. The planar case (i). In Figure 1 we indicate a method of numbering all the squares of a plane lattice. To continue from where the numbering stops one just works round and round the diamond taking care at the corners. For each integer $m \geqq 0$ let S_m be the diamond-shaped set of squares of Figure 1 having numbers $\leqq m$. It will appear that for all finite sets S of $m \geqq 1$ squares

(1) $$|B(S)| \geqq |B(S_m)|$$

and

(2) $$|B(S)|^2 > 8|S|.$$

In establishing (1) and (2) we will assume S is "connected", for otherwise we translate the "components" of S until they "touch" and $|B(S)|$ goes down in the process. Consider the case where S is the shaded region of Figure 2. The rectangle $abcd$ is the smallest rectangle which contains S and has its sides along diagonal lines of the lattice. Hence on each of its sides there is a point at which it meets S; let them be w, x, y, z as shown. Because S is connected, in each of the horizontal rows of squares lying between w and x, there is a square of $B(S)$ lying adjacent to and to the right of a square of S. Also in each of the horizontal rows between y and z there is a square of $B(S)$ to the left of a square of S. Then we get the corresponding result for the columns between x and y, and for the columns between z and w. Thus $|B(S)|$ is greater than or equal to the number of squares containing segments of $abcd$. Clearly $|S|$ is not greater than the number of squares inside $abcd$, and the argument is completely general. Inequalities (1) and (2) now follow easily, though various details need checking. Moreover it is clear that we will get strict inequality in (1) if S is not almost diamond-shaped, though the precise condition is rather more involved, for example a square of 4 squares has the same number of boundary squares as S_4.

The number 8 in inequality (2) is the best possible. This is shown by the regular diamond S_m with $m = 1 + 2n(n+1)$ and $|B(S)| = 4(n+1)$ for an arbitrarily large integer n.

3. The three dimensional case (ii). Here we will give a heuristic argument using Steiner symmetrization which suggests that for all finite nonempty sets S,

$$|B(S)|^3 > 36|S|^2, \tag{3}$$

and that S is approximately an octahedron in the limiting case. That the number 36 in (3) cannot be reduced is shown by the fact that $1 + 2n + \frac{2}{3}n(n+1)(2n+1)$ cubes formed into a "regular" octahedron have a boundary of $2 + 4(n+1)^2$ cubes.

Given S we consider it as being composed of layers of cubes, each layer lying between two planes which are perpendicular to one of the lattice coordinate directions. Then if a given layer contains m cubes we rearrange them in the shape of the planar diamond S_m. Moreover the rearrangement of all the layers is such that the diamonds have the same orientation and their centers lie one above another. We claim that this rearrangement does not increase $|B(S)|$. Firstly the number of boundary cubes of a layer which lie in the layer is minimal after rearrangement by (1). Next we note that the number of boundary cubes in a layer which are needed to bound the cubes of S in an adjacent layer is also as small as possible after rearrangement, because arranging the diamonds to lie on top of one another causes the cubes in one layer to "cover up" for the cubes in the adjacent layer as much as possible. Moreover if any one layer of S is not diamond-shaped then the rearrangement causes a strict reduction in $|B(S)|$. Taking S to be layered in each of the three possible ways one after another repeatedly, and rearranging each time, will clearly cause S to end up octahedron-shaped. It would require a great deal of effort to examine every detail of every case, so we will not try. However (3) certainly holds for a regular octahedron, and for small values of $|S|$, and is believed to hold in general.

THE ORTHIC TRIANGLE AND AN INEQUALITY OF EULER

G. D. CHAKERIAN, University of California, Davis

The orthic triangle of a given acute angled triangle ABC is that triangle whose vertices are the feet of the altitudes of ABC. It is a well-known result, originating with J.F. de Fuschis a Fagnano in 1775, that of all triangles inscribed in ABC, the orthic triangle has the least perimeter. This result has received several elegant proofs, among them a proof of H. A. Schwarz, using mirror reflections, and an ingenious proof of L. Fejér (see [**5**] for these proofs).

The properties of the orthic triangle have been extensively studied, and one finds a thorough treatment in Johnson [**4**]. A particular property that will be of interest in this note is the following, found on page 191 of [**4**], and also proved on page 86 of [**1**]. Let Δ be the area of our triangle, let R be the radius of the circumscribed circle and p the perimeter of the orthic triangle. Then

(1) $$p = \frac{2\Delta}{R}.$$

If r is the radius of the inscribed circle and P the perimeter of ABC, then $2\Delta = rP$. Applying this in (1) we obtain

(2) $$\frac{p}{P} = \frac{r}{R}.$$

Now observe that the triangle determined by the midpoints of the sides of ABC has perimeter $P/2$. Since this midpoint triangle is a particular triangle inscribed in ABC, the minimizing property of the orthic triangle implies

(3) $$p \leqq \frac{P}{2}.$$

This inequality, in conjunction with (2), yields a famous inequality of Euler,

(4) $$R \geqq 2r,$$

an inequality that has received considerable attention.

A very simple and beautiful proof of (4) was given by I. Ádám and is reproduced in the book of L. Fejes Tóth [**3**, p. 28]. Our derivation of (4) is not very satisfactory, depending as it does ultimately on (1). But we thought it of interest to direct attention to (2), from which follows the equivalence of (3) and (4). Furthermore, the simple justification of (3) appears to have been overlooked by various authors in the past. Relatively complicated proofs of (3) are given in [**2**], [**6**], and [**1**, p. 86]. Those proofs proceed in a direct manner, without invoking the minimizing property of the orthic triangle. In [**1**], the inequality (3) is obtained by first deriving (1) and then applying (4).

Equality holds in (4) if and only if it holds in (3). In this case the orthic triangle must coincide with the midpoint triangle, which can happen only if ABC is equilateral.

Finally, note that although our derivation of (4) was restricted to acute triangles, if ABC has an obtuse angle then the center of the circumscribed circle is outside the inscribed circle, so clearly $R > 2r$.

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A CURIOUS PROPERTY OF THE INTEGER 38

ERWIN JUST and NORMAN SCHAUMBERGER, Bronx Community College, CUNY

Goldbach's unproved conjecture asserts that any even integer greater than 2 can be expressed as the sum of two primes. We ask a question which is similar in appearance. Can any positive even integer be expressed as the sum of two positive odd composites? We observe immediately that the answer must be given in the negative. The set of positive even integers $\leqq 38$ which cannot be expressed as the sum of two positive odd composites is $\{4, 6, 8, 12, 14, 20, 32, 38\}$. Are these 8 integers the only even integers which cannot be expressed as the sum of two positive odd composites or does there exist an even integer greater than 38 with this property? We can prove quickly that 38 is the largest even integer which cannot be expressed as the sum of two positive odd composites. To this end, let n denote an even integer greater than 38 and note that n must have the form $10k$, $10k+2$, $10k+4$, $10k+6$ or $10k+8$. To show that, in each of these cases, n can be expressed as a sum of two positive odd composites, it suffices to observe that

(1) When $n = 10k$, then $n = 15 + (10k - 15)$,
(2) When $n = 10k + 2$, then $n = 27 + (10k - 25)$,
(3) When $n = 10k + 4$, then $n = 9 + (10k - 5)$,
(4) When $n = 10k + 6$, then $n = 21 + (10k - 15)$ and
(5) When $n = 10k + 8$, then $n = 33 + (10k - 25)$.

Since, in each case, we have succeeded in expressing n as a sum of two positive odd composites, we have established the rather surprising fact that there are precisely 8 positive even integers which cannot be so expressed and of these, 38 is the largest.

IF n LINES IN THE EUCLIDEAN PLANE MEET IN 2 POINTS THEN THEY MEET IN AT LEAST $n-1$ POINTS

FRANK PAVLICK, Slippery Rock, Pennsylvania

A number of articles concerning Sylvester's problem [5] and its relatives have appeared in this and other journals. None have mentioned the little result in the title which is amazingly simple to obtain. For small n the proposition is easily verified by a process of elimination of configurations and, in fact, is trivially true for $n \leqq 3$. However this approach is not applicable from a practical standpoint for large n in proving the proposition in its generality.

It was proven by Melchior [4] that if n lines are not all parallel and do not meet in a common point, then there exists a point that lies on exactly two of the lines. Let us call such a point an *ordinary point*. Although there is additional information on the minimum number of such points in a configuration of n lines, Melchior's result, along with induction, is sufficient for proving the proposition.

A CURIOUS PROPERTY OF THE INTEGER 38

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(3) When $n = 10k + 4$, then $n = 9 + (10k - 5)$,
(4) When $n = 10k + 6$, then $n = 21 + (10k - 15)$ and
(5) When $n = 10k + 8$, then $n = 33 + (10k - 25)$.

Since, in each case, we have succeeded in expressing n as a sum of two positive odd composites, we have established the rather surprising fact that there are precisely 8 positive even integers which cannot be so expressed and of these, 38 is the largest.

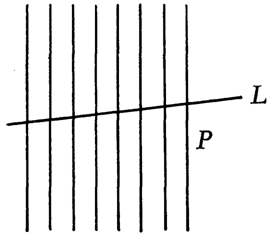
IF n LINES IN THE EUCLIDEAN PLANE MEET IN 2 POINTS THEN THEY MEET IN AT LEAST $n - 1$ POINTS

FRANK PAVLICK, Slippery Rock, Pennsylvania

A number of articles concerning Sylvester's problem [5] and its relatives have appeared in this and other journals. None have mentioned the little result in the title which is amazingly simple to obtain. For small n the proposition is easily verified by a process of elimination of configurations and, in fact, is trivially true for $n \leqq 3$. However this approach is not applicable from a practical standpoint for large n in proving the proposition in its generality.

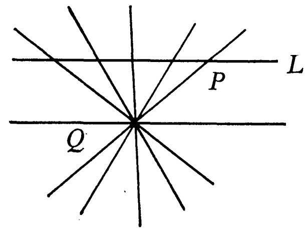
It was proven by Melchior [4] that if n lines are not all parallel and do not meet in a common point, then there exists a point that lies on exactly two of the lines. Let us call such a point an *ordinary point*. Although there is additional information on the minimum number of such points in a configuration of n lines, Melchior's result, along with induction, is sufficient for proving the proposition.

Proof. Suppose the proposition is true for $n-1$ lines and let K be a configuration of n lines meeting in at least two distinct points. According to Melchior, there is at least one ordinary point P. Let L be one of the two lines passing through P. Taking away L eliminates P and leaves a configuration K' consisting of $n-1$ lines. These lines are parallel, or meet in one point, or by the induction hypothesis meet in at least $n-2$ points.



$n-1$ parallel lines

FIG. 1.



$n-1$ lines intersecting at Q

FIG. 2.

Case 1. If all $n-1$ lines in K' are parallel, then L must have intersected each of the lines, resulting in $n-1$ distinct points in K. See Figure 1.

Case 2. If all $n-1$ lines in K' intersect at a common point Q, then L could have been parallel to at most one of them, intersecting the others in $n-2$ distinct points. Adding Q results in $n-1$ points in K. See Figure 2.

Case 3. If the $n-1$ lines intersect in $n-2$ or more points, with the addition of P, the resulting number of points in K is at least $n-1$.

This leads to a further conjecture. After investigation of the cases where $n = 4, 5, 6$, and 7, one could construct the following table:

Number of Lines	Possible Numbers of Points of Intersection
4	$0, 1, 3 = 4-1, 4, 5, 6 = C(4,2)$
5	$0, 1, 4 = 5-1, 5, 6, 7, 8, 9, 10 = C(5,2)$
6	$0, 1, 5 = 6-1, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 = C(6,2)$
7	$0, 1, 6 = 7-1, 7, 8, 9, 10, \cdots, 21 = C(7,2)$

This leads us to believe that the following is true:

CONJECTURE. There exists a configuration in the Euclidean plane consisting of n lines meeting in p points where p is 0, 1 or any integer satisfying $n-1 \leqq p \leqq C(n,2)$.

Some of the numbers of intersection points are easily verified; e.g., $n-1$, n, and $C(n,2)$. However, the author has never seen the proof or denial of the conjecture in print. Further results relating to Sylvester's problem may prove to be beneficial. For instance it has been shown that the number of ordinary points is at least $3n/7$.

For excellent bibliographies on these results see [**1**, **2**, **3**].

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ANALYTIC FUNCTIONS ON NONOPEN SETS

CARL DAVID MINDA, University of Cincinnati

First we recall the customary definition of an analytic function whose domain is an open set in the complex plane $\mathbb{C}$, [**1**, p. 36]. Given a point $a \in \mathbb{C}$, a neighborhood N of a and $f: N \to \mathbb{C}$, we say that f is analytic at a if there is a power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ having positive radius of convergence whose sum is equal to f in some neighborhood of a. Given an open set U in $\mathbb{C}$ and $f: U \to \mathbb{C}$, we say that f is analytic on U if f is analytic at each point of U. It is well known that if N is a neighborhood of a and $f: N \to \mathbb{C}$ is analytic at a, then there is an open neighborhood V of a, $V \subset N$, such that $f|V$ is analytic on V [**1**, p. 37].

For an arbitrary set $A \subset \mathbb{C}$ there is a local and a global way to define analyticity.

DEFINITION. *Let $A \subset \mathbb{C}$ and $f: A \to \mathbb{C}$. (i) f is said to be analytic on A if there is an open set $U \supset A$ and a function F analytic on U such that $F|A = f$.*

(ii) f is locally analytic on A if for every $a \in A$ there is a neighborhood N_a of a and a function $F_a: N_a \to \mathbb{C}$ analytic at a such that F_a and f agree on $N_a \cap A$.

REMARKS. Part (i) is the usual way in which the concept of analyticity is defined for a function $f: A \to \mathbb{C}$. However, (ii) is analogous to the manner in which the concept of analyticity is defined for the case in which A is open and might seem more natural. Note that (ii) is equivalent to requiring that there is an open neighborhood V_a of a and a function F_a analytic on V_a such that $F_a|(V_a \cap A) = f|(V_a \cap A)$. It is clear that if f is analytic on A, then f is locally analytic on A. We shall establish the converse.

LEMMA. *Let A be a set in $\mathbb{C}$ and let A' denote the set of accumulation points of A. If W is an open set in $\mathbb{C}$ such that $A \cap A' \subset W$, then there is an open set $V \subset W$ such that $A \cap A' \subset V$ and $A \cap \partial V = \varnothing$.*

Proof. If $A \cap \partial W = \varnothing$, simply take $V = W$. Henceforth we shall assume that $A \cap \partial W \neq \varnothing$. Since $A = (A \cap A') \cup (A \sim A')$ and $A \cap A' \subset W$, it follows that $A \cap \partial W = (A \sim A') \cap \partial W$. Note that the set $A \sim A'$ is countable since every point of $A \sim A'$ is isolated. Also, for every $a \in A \sim A'$ there is a $\rho_a > 0$ such that the sets $B(a, \rho_a) = \{z \in \mathbb{C}: |z - a| < \rho_a\}$ are pairwise disjoint and $\overline{B(a, \rho_a)} \cap A = \{a\}$. Let $(A \sim A') \cap \partial W = \{a_i: i \in I\}$ where the index set I is either finite or $I = \mathbb{N}$, the natural numbers. In the latter case we select $\rho_i = \rho_{a_i}$ so that $\rho_i \to 0$ as $i \to \infty$. Set $B_i = B(a_i, \rho_i)$, $B = \bigcup_{i \in I} B_i$ and $V = W \sim \bar{B}$, then V is an open set.

Now we show that $a \in A$ implies that $a \notin \partial B$. If $a \in A$ and $a \in \partial B$, then there is a sequence $(z_n)_{n=0}^{\infty}$ in B with $z_n \to a$. If there is an $i \in I$ such that $z_n \in B_i$ for infinitely many values of n, then $a \in \bar{B}_i$ which implies that $a = a_i$. This is impossible because a_i is an interior point of B. Otherwise, I must be infinite and each B_i contains just a finite number of terms from the sequence. Then we can extract a subsequence $(z_{n_j})_{j=0}^{\infty}$ such that $z_{n_j} \in B_{i_j}$ and $i_j < i_{j+1}$. Since $\rho_i \to 0$, this implies that $z_{n_j} - a_{i_j} \to 0$ and $a_{i_j} \to a$. Hence, $a \in A'$. Since all a_i belong to the closed set ∂W, $a \in \partial W$. Therefore, $a \in \partial W \cap A \cap A' = \varnothing$. This contradiction proves that $A \cap \partial B = \varnothing$.

Consequently, $A \cap \bar{B} = \{a_i: i \in I\}$ and this guarantees that $A \cap A' \subset V$. All that remains is to show that $A \cap \partial V = \varnothing$. From $\partial V \subset \partial W \cup \partial B$ and $A \cap \partial B = \varnothing$, we obtain $A \cap \partial V \subset \{a_i: i \in I\}$. But each $a_i \in B$ so $a_i \notin \partial V$. Thus, $A \cap \partial V = \varnothing$.

PROPOSITION. *Let $A \subset \mathbb{C}$ and $f: A \to \mathbb{C}$. If f is locally analytic on A, then f is analytic on A.*

Proof. Since f is locally analytic on A, for every $a \in A$ there is an $r_a > 0$ and a function F_a analytic on $B(a, r_a)$ such that $F_a(z) = f(z)$ for all $z \in B(a, r_a) \cap A$. Set $W = \bigcup_{a \in A \cap A'} B(a, \frac{1}{2}r_a)$, then W is an open set and $A \cap A' \subset W$. Define a function $G: W \to \mathbb{C}$ as follows: For each $z \in W$ there is an $a \in A \cap A'$ with $z \in B(a, \frac{1}{2}r_a)$; set $G(z) = F_a(z)$. The function G is well defined. Suppose that $a, b \in A \cap A'$ and $z \in B(a, \frac{1}{2}r_a) \cap B(b, \frac{1}{2}r_b)$. Without loss of generality we may assume that $r_a \leqq r_b$. In this situation $a \in B(b, r_b)$. Since $a \in A'$ and F_a and F_b agree on A in a neighborhood of a, it follows from the identity theorem for analytic functions [**1**, p. 40] that F_a and F_b coincide on $B(a, r_a) \cap B(b, r_b)$. In particular, $F_a(z) = F_b(z)$ so that G is well defined. Next, G is analytic on W. Let $z \in W$, then there is an $a \in A \cap A'$ with $z \in B(a, \frac{1}{2}r_a)$ and $F_a(z) = G(z)$. Clearly, $G | B(a, \frac{1}{2}r_a) = F_a$ so G is analytic at z. Also, $G | (W \cap A) = f | (W \cap A)$.

By employing the lemma we can find an open set V with $A \cap A' \subset V \subset W$ and $A \cap \partial V = \varnothing$. Every point of A lies in either V or $\mathbb{C} \sim \bar{V}$ and each point of A in $\mathbb{C} \sim \bar{V}$ is an isolated point of A. Since $\mathbb{C} \sim \bar{V}$ is open, for each $a \in A \cap (\mathbb{C} \sim \bar{V})$ we can determine $R_a > 0$ such that $B(a, R_a) \subset \mathbb{C} \sim \bar{V}$ and the sets $B(a, R_a)$ are pairwise disjoint. For each $a \in A \cap (\mathbb{C} \sim \bar{V})$ find F_a analytic on $B(a, R_a)$ such that $F_a(a) = f(a)$. Let $U = V \cup \bigcup_{a \in A \cap (\mathbb{C} \sim \bar{V})} B(a, R_a)$ and define $F: U \to \mathbb{C}$ by $F | V = G | V$ and $F | B(a, R_a) = F_a$ for $a \in A \cap (\mathbb{C} \sim \bar{V})$, then F is analytic on U and $F | (U \cap A) = f$.

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A SIMPLE CONSTRUCTION OF A NON-DESARGUESIAN PLANE

SUBHASH C. SAXENA, The University of Akron

In a course in projective geometry one runs into non-Desarguesian planes. One of the most frequently used examples is that of a Moulton plane [1]. Another given in a few books [2, 3] is that of a free projective plane. The purpose of this note is to discuss the last example and present a simpler proof than the ones given in the books.

Let A_1 be a complete quadrangle consisting of four distinct points P_1, P_2, P_3, P_4 (no three of them being collinear) and the six distinct lines determined by these four points. We write

$$A_1 = \{P_1, P_2, P_3, P_4\} \cup \{P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, P_3P_4\}.$$

Next, construct the set A_2 which contains all the points and lines of A_1 and the additional distinct points obtained from those pairs of lines of A_1 which do not have any point of A_1 in common. We write

$$P_5 = P_1P_2 \cdot P_3P_4,$$

$$P_6 = P_1P_3 \cdot P_2P_4,$$

$$P_7 = P_1P_4 \cdot P_2P_3.$$

Let A_3 contain all the points and lines of A_2 and the additional distinct lines P_5P_6, P_5P_7, P_6P_7 determined by those pairs of points of A_2 which are not on a line in A_2.

Next, A_4 is constructed by taking all the points and lines of A_3 and the additional distinct points $P_8 = P_2P_3 \cdot P_5P_6$, $P_9 = P_2P_4 \cdot P_5P_7$, $P_{10} = P_3P_4 \cdot P_6P_7$.

A_5 will then contain the points and lines of A_4 and the additional lines P_8P_9, P_9P_{10}, $P_{10}P_8$ (all distinct).

In general, we let A_{2n} contain the points and lines of A_{2n-1} and the additional points obtained from those pairs of lines of A_{2n-1} which do not have any point of A_{2n-1} in common, and we use the dual of it to obtain A_{2n+1}.

Now let $\pi = \bigcup_{n=1}^{\infty} A_n$.

It is obvious that π satisfies the incidence (and existence) postulates of a rudimentary projective plane [2]. However, if we consider the triangles $P_2P_3P_4$, $P_5P_6P_7$, they are perspective from the point P_1, whereas P_8, P_9, P_{10}, which are the points of intersection of the corresponding sides of these triangles, are not collinear. Hence π is a non-Desarguesian plane.

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A NOTE ON MATRIX INVERSION

A. POLTER GEIST, Miskatonic University, Arkham, Mass.

In discussing certain stochastic processes, it is useful to know the inverse of the n-by-n matrix S whose entries on the main diagonal are 0, and all of whose other entries are 1. The following method may be of interest.

Let U be the n-by-n matrix all of whose entries are 1. Then, $S = U - I$. Thus,

$$-S^{-1} = (I - U)^{-1} = I + U + U^2 + U^3 + \cdots.$$

It is easily seen that $U^2 = nU$, $U^3 = n^2U$, etc., so that we have

$$\begin{aligned} -S^{-1} &= I + U + nU + n^2U + \cdots \\ &= I + U(1 + n + n^2 + \cdots) \\ &= I + U\frac{1}{1-n}, \end{aligned}$$

and we obtain the desired inverse

$$S^{-1} = \frac{1}{n-1}U - I.$$

It is clear that this fails when $n = 1$.

NOTES AND COMMENTS

Joseph S. Madachy writes regarding Michael Deakin's *Walking in the rain* in the November 1972 issue that a similar article *A walk in the rain* by Alan Sutcliffe appeared in the February 1962 issue of Recreational Mathematics Magazine.

Andrzej Makowski notes that inequalities (f), (g), (h), (k), (l), (m), and (n) of the paper *An approach to trigonometric inequalities* by Harold Ehret (this MAGAZINE, 43 (1970) 254-257) were also obtained from the Erdös inequality by J. M. Child in his paper *Inequalities connected with a triangle* (Math. Gazette, 23 (1939) 138–143; see especially p. 142).

From Roberto Frucht regarding *The distribution of quadratic residues in fields of order* p^2 by Bergum and Jordan in the September 1972 issue: "It seems that the authors were not aware of the paper by Reinaldo E. Giudici *Quadratic residues in* $GF(p^2)$, this MAGAZINE, 43 (1971) 153-157. Indeed the authors study in their paper only the special case where p is a prime of the form $8k + 5$; Giudici however had already considered in his paper the general case of odd primes without any restriction. As a consequence most of the results obtained by Bergum and Jordan can be read off immediately from Giudici's paper by letting $p \equiv 5 \pmod 8$ and $g = -2$. Compare for instance Bergum-Jordan's Theorem 4.1 with Giudici's Theorem 5."

Editor's comment. "The Giudici paper was first submitted to this MAGAZINE in October 1970. The Bergum and Jordan paper was first submitted to this MAGAZINE in March 1969. It was then lost in the editorial offices of this MAGAZINE and it was eventually published in the September 1972 issue. The decision to publish it may have been wrong but Professors Bergum and Jordan do have a legitimate claim to priority. The editor was remiss in not explaining this at the time."

BOOK REVIEWS

EDITED BY ADA PELUSO AND WILLIAM WOOTON

Materials intended for review should be sent to: Professor Ada Peluso, Department of Mathematics, Hunter College of CUNY, 695 Park Avenue, New York, New York, 10021, or to Professor William Wooton, 1495 La Linda Drive, Lake San Marcos, California 92069. A boldface capital **C** *in the margin indicates that a review is based in part on classroom use.*

Trigonometry: Circular Functions and Their Applications. By James E. Hall. Brooks/Cole Publishing Company, Monterey, California, 1973. 222 pp., $7.95.

In a field where the books are numerous, it is always pleasant to encounter the rare one that can present the material succinctly and in an interesting manner. This book has captured that spirit, and, in a brief 183 pages, deals with the traditional topics of trigonometry, developed first from the unit circle, then extended to real-valued functions on angles. All this is built upon a brief but thorough review of the algebraic concepts needed in the balance of the development. The book maintains, as the author states, "a middle course on rigor, blending informal motivation with formal definition and development."

Figures, examples and exercises all seem to be well chosen. Answers to the odd-numbered exercises are included, but no graphs are given in the answer section.

JAMES HARDESTY, Los Angeles Pierce College

The Fascinating World of Mathematics. By J. N. Kapur. S. Chand and Co., New Delhi, India, 1970. 269 pp., Rs. 16.50.

This book is a collection of lectures and addresses given by the author to various Indian learned societies, summer institutes, universities, etc. The book is divided into three sections titled (I) Nature of Mathematics, (II) Applications of Mathematics, and (III) Mathematics Education.

Typographical errors are so numerous in the book, particularly in the first section, that some individual articles would be virtually incomprehensible to a reader who was not well acquainted with the topic.

The topics of Section I are those which are normally discussed in such courses as "Mathematics for Liberal Arts"; however, the book could hardly be used as a text for such a course.

Most readers will find Section II to be of value. Although details are not presented, the articles of this section describe or allude to applications of mathematics with which most students and professors are not acquainted. Topics include discussions of applications in biomedical sciences, coding theory, quantum mechanics, weaponry, economics, operations research, etc.

The final section is concerned with mathematics education in India. The topics in this section range from a plea for changing the names of the integers between ten and twenty to pleas for an Indian revolution in mathematics curricula and teaching.

TERRAL MCKELLIPS, Cameron College, Lawton, Oklahoma

Why Johnny Can't Add: The Failure of the New Math. By Morris Kline. St. Martin's Press, New York, 1973. 173 pp., $6.95.

The title, reminiscent of the earlier text *Why Johnny Can't Read* by Rudolph Flesch, obscures the fact that this is a book for the mathematician as well as for the school principal. Professor Kline presents a bold, comprehensive, and long overdue critique of the modern mathematics curricula for the elementary and secondary schools. Among the various groups of teachers, supervisors, publishers, etc., that are responsible for the development of the new curricula, the mathematicians are the main target of Kline's criticism: "The professional mathematicians are the most serious threat to the life of mathematics, at least so far as the teaching of the subject is concerned."

The first five chapters deal with a critical evaluation of the traditional and the new mathematics curricula. The author argues against the rigorous deductive approach advocated by the so called "modernists." Through a historical analysis, an area in which Kline's knowledgeability is evidenced by his monumental text *Mathematical Thought from Ancient to Modern Times*, he shows that mathematical creations were achieved intuitively and not logically. It is argued that an axiomatic approach to teaching mathematics gives the pupils the impression that the subject is created by "geniuses" who arrive at theorems directly from the axioms. Since the pupil cannot, beyond memorization, duplicate this process he often feels defeated.

The three subsequent chapters deal with subject matter. The merits of introducing concepts in set theory, Boolean algebra, congruences, number system bases, inequalities, and symbolic logic are questioned as to their usefulness for rendering the essence of contemporary mathematics. Furthermore, their applicability for the nonspecialist is deemed doubtful. Chapter nine points to the lack of empirical evidence that would favor the new curricula to the traditional ones. Chapter ten speculates on the reasons for the new programs. These recent endeavors are considered lacking in motivation. The last chapter, therefore, outlines the direction of future reforms.

The book is well documented. It is lucid, humorous, and its many analogies and excursus enhance its readability. The author obviously aims at generating controversy.

He thus leaves himself open to criticism. In his "proper direction for reform" he does not offer novel solutions. For example, the mathematics laboratory which is here advocated has for some time been an essential component of the modern curricula. Illustrative examples dealing with visual representations of algebraic equations, heuristic and physical arguments, and analogies do not open new vistas beyond the already existing curricula. His ideas concerning the further development of laboratory materials which would necessitate the collaborative efforts of teachers and engineers and for the use of oscilloscopes in the teaching of trigonometry are unrealistic. Most schools cannot even afford to buy a sufficient quantity of Cuisenaire rods.

The polemic stance of the author may provoke most rigorous disagreements with his philosophy of mathematics education. This should not, however, deter individuals who are involved in the mathematical preparation of elementary and secondary school teachers. The book will serve as a timely impetus for the re-evaluation of the problems that face Johnny.

JOHN NIMAN, Hunter College, CUNY

ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1973 recipients of these Awards, selected by a committee consisting of E. F. Beckenbach, Chairman, Marvin Marcus, and D. E. Richmond, were announced by President R.P. Boas at the business meeting of the Association on August 21, 1973, at the University of Montana. The recipients of the Ford Awards for articles published in 1972 were the following:

J. A. Dieudonné, The Historical Development of Algebraic Geometry, MONTHLY, 79 (1972) 827–866.

Samuel Karlin, Some Mathematical Models of Population Genetics, MONTHLY, 79 (1972) 699–739.

P. D. Lax, The Formation and Decay of Shock Waves, MONTHLY, 79 (1972) 227–241.

T. L. Saaty, Thirteen Colorful Variations on Guthrie's Four-Color Conjecture, MONTHLY, 79 (1972) 2–43.

L. A. Steen, Conjectures and Counterexamples in Metrization Theory, MONTHLY, 79 (1972) 113–132.

R. L. Wilder, History in the Mathematics Curriculum: Its Status, Quality, and Function, MONTHLY, 79 (1972) 479–495.

HENRY L. ALDER, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before January 1, 1974.

PROPOSALS

873. *Proposed by A. G. Bradbury, North Bay, Ontario, Canada.*

In this alphametic, of course, each distinct letter stands for a particular but different digit in the decimal notation. The array of three-digit words corresponds to a rectangular Magic Square. What must the car be?

$$\begin{array}{r} A \\ N\ E\ W \\ \underline{T\ O\ Y} \\ C\ A\ R \end{array}$$

874. *Proposed by David Singmaster, London, England.*

An $n \times n$ array is called supermagic if every $n-1 \times n-1$ subarray obtained by removing a column and a row has the same sum. Find all supermagic squares.

875. *Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.*

If $\{a_i\}$, $\{b_i\}$ denote two sequences of positive numbers and n is a positive integer, show that:

$$\sum_i a_i^{2n} \cdot \sum_j b_j^{2n} \geqq \sum_i a_i^{2n-1} b_i \cdot \sum_j a_j b_j^{2n-1} \geqq \cdots \geqq \sum_i a_i^n b_i^n \cdot \sum_j a_j^n b_j^n .$$

876. *Proposed by Steven R. Conrad, Benjamin Cardozo High School, Bayside, New York.*

Let $\{a_i\}$ represent a set of n arbitrary real numbers whose sum is positive. Prove that no matter how these numbers are arranged on the circumference of a circle, there always exists at least one of them, to be called a_1, such that if these numbers are subscripted consecutively from 1 to n in a clockwise direction, it is always true that

$$\sum_{i=1}^{k} a_i > 0 \text{ for } k = 1, 2, \cdots n.$$

877. *Proposed by H. J. Webb, Cape Town University, Republic of South Africa.*

In triangle ABC, $\angle ABC = \angle ACB = 40°$. AB is produced to D such that $AD = BC$. Determine $\angle BCD$.

878. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New York.*

$$\text{Prove } 1 + x > \left(1 + \frac{x}{p^2 - 1}\right)^p \text{ when } 0 < x < 2^p - 1.$$

879. *Proposed by Sally Ringland, Clarion State College, Pennsylvania.*

Given ΔABC, let P be any point on AB, Q be any point on BC and R any point on AC. P, Q and R should not be vertices of the triangle. Consider the circles through A, P and R, through P, B and Q and through Q, C and R. Prove that the centers of the circles determine a triangle similar to ΔABC.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q573. An earlier issue of this MAGAZINE contained the solution to the problem of finding the dimensions of all rectangles whose area and perimeter are the same positive integer, i.e.,

$$\text{Length} = \frac{k + \sqrt{k^2 + 16k}}{4}, \quad \text{Width} = \frac{k - \sqrt{k^2 + 16k}}{4},$$

where k is a positive integer. Without using calculus, find the limit of the width as $k \to \infty$.

[*Submitted by Robert J. Rapalje*]

Q574. Defining $C(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$ and $S(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$ show that $C^2(x) + S^2(x) = 1$ without the obvious identification of $C(x)$ and $S(x)$.

[*Submitted by C. F. Pinzka*]

Q575. One hour after leaving a station, a train broke down and had to continue the trip at 3/5 of its original speed. The train arrived at the next station 2 hours late. If the mishap had occurred 50 miles further on the train would have arrived 40 minutes sooner. How fast was the train originally traveling?

[*Submitted by E. F. Schmeichel*]

Q576. Given an n-dimensional simplex $OA_1A_2 \cdots A_n$ whose edges emanating from O are mutually orthogonal. Show that the square of the content of the $(n-1)$-dimensional face opposite O is equal to the sum of the squares of the contents of the remaining faces.

[*Submitted by Murray S. Klamkin*]

Q577. Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$ in which the a_i are integers and the domain of f is the set of positive integers. Prove that a_i can be chosen so as to guarantee that the range of f will contain at least n distinct primes.

[*Submitted by Erwin Just and Howard Kleiman*]

(Answers on page 240.)

Errata. The name of *Steven Diaz, Bishop Feehan High School, Attleboro, Massachusetts,* was omitted from the list of Also Solvers of Problem **829.**

In Problem **871**, May 1973, Page 167, in the second line the letter G should read H.

SOLUTIONS

Magic Square Determinants

845. [November, 1972] *Proposed by Richard L. Breisch, Pennsylvania State University.*

Do there exist two 3×3 magic squares A and B, each containing the digits 1 through 9 such that

$$\det A + \det B = \det(A + B)?$$

I. *Solution by Marjorie Fitting, California State University at San Jose.*

Yes. Let A be any magic square. Form B by reflecting A across the vertical (or horizontal) center column (row). Then $\det A = -\det B$ since the diagonals have reversed roles, $A + B$ will have the first and third columns identical, and

$$\det A + \det B = \det A + -\det A = 0 = \det(A + B).$$

II. *Solution by Zalman Usiskin, University of Chicago.*

The answer is "Yes", and a much stronger result is easily proved. Let A be *any*

$n \times n$ magic square where $n = 4k + 3$. Let B be the magic square formed from A by reversing the order of the columns in A. Since $n = 4k + 3$, there are $2k + 1$ columns to the left of the middle column which have been transposed with $2k+1$ columns to the right of the middle column. The odd number of transpositions implies $\det B = -\det A$. So $\det A + \det B = 0$.

But the first and last columns of $A + B$, each being the sum of the first and last columns of A, are identical. Thus $\det(A + B) = 0 = \det A + \det B$.

A similar argument is possible when $n = 4k + 2$.

Also solved by Charles N. Baker, West Liberty State College, West Virginia; Gladwin E. Bartel, Otero Junior College, Colorado; Marjorie Bicknell, Wilcox High School, Santa Clara, California; M. T. Bird, California State University, San Jose; Eliot W. Collins, Brockport, New York; Romae J. Cormier, Northern Illinois University; Clayton W. Dodge, University of Maine at Orono; Rosalie Garrand, Palo Alto, California; Richard A. Gibbs, Fort Lewis College, Colorado; Michael Goldberg, Washington, D. C.; Alvin A. Gloor, Westside High School, Omaha, Nebraska; Thomas W. Hill, Jr., Purdue University; Ralph Jones, University of Massachusetts; D. L. Muench, St. John Fisher College, New York; Francis D. Parker, St. Lawrence University; Donald E. Rossi, De Anza College, California; Sally Ringland, Clarion State College, Pennsylvania; E. F. Schmeichel, Itasca, Illinois; James Tattersall, Attleboro, Massachusetts; Charles W. Trigg, San Diego, California; Mary Frances Turner, Mathematics and Science Center, Richmond, Virginia; and the proposer.

The Streets of Paris

846. [November, 1972] *Proposed by Alfred Kohler, Long Island University, New York.*

Baron Georges Hausmann has decided to reconstruct the streets of Paris so that for every set of three important points in the city which happen to lie on a straight line, a boulevard joining these points is to be constructed. The treasury and the post office are on Rue Madrid, the opera house and police headquarters are on Rue Roma, and the railroad terminal and palace are on Rue Berlin: all three of these streets radiate from Étoile in straight lines. The Baron has already ordered the construction of boulevards joining the treasury, the opera house and the theater; the post office, the police headquarters and the theater; the palace, the police headquarters and the cathedral; the railroad terminal, the treasury and the food market; and the palace, the post office and the food market.

The Baron now intends to issue an order for the construction of a boulevard joining the food market, the cathedral and the theater since he thinks that these three points also lie on a straight line. The owners of the houses lying along the proposed route are opposed to the new boulevard, and they maintain that these points do not lie along a straight line after all. Resolve this dispute by determining whether or not the food market, the cathedral and the theater do lie along a straight line.

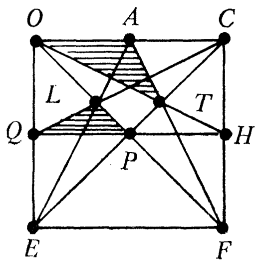
Editor's Note: One phrase was omitted from the statement of the problem. That statement placed the railroad terminal, the opera house, and the cathedral as also being collinear. The solution below used this fact.

Solution by Clayton W. Dodge, University of Maine at Orono.

Denote the points in order of mention by T, P, O, Q, A, L, E, H, C, and F. In the stated problem, C is specified only as lying on line LQ, and it is easy to draw a figure satisfying all the given conditions. Then lines FH and LQ intersect (or they can be drawn parallel), so C may or may not be taken as their point of intersection.

If we require also that the cathedral, the railroad terminal, and the opera house (C, A, and O) are collinear, then F, C, and H are collinear by Desargues' two-triangle theorem since triangles OAT and QLP are copolar at E, whence they are also coaxial on the line CHF.

One figure satisfying all the above data is a square $COEF$ in which H, A, Q are the midpoints of sides FC, CO, OE, and P is the midpoint of QH. Let L and T be the centroids of triangles OEC and OFC. Note that we also have the lines $PECT$ and $FLOP$.



Editor's Note. *The following persons either pointed out the omitted information and supplied it, thereby proving collinearity, or used the given information and proved that the three points could not be proven collinear: Gladwin E. Bartel, Otero Junior College, Colorado; R. X. Brennan ,Dover, New Jersey; Maxey Brooke, Sweeny, Texas; Joseph A. Capalbo, Bradford, Rhode Island; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Howard K. Hilton, Prosser Vocational High School, Chicago, Illinois; Lew Kowarski, Morgan State College, Maryland; C. F. Pinzka, University of Cincinnati; E. F. Schmeichel, Itasca, Illinois; Benjamin L. Schwartz, McLean, Virginia; James Tattersall, Attleboro, Massachusetts; Charles W. Trigg, San Diego, California; and the proposer.*

An Exponential Inequality

847. [November, 1972] *Proposed by R. Shantaram, University of Michigan-Flint.*

Prove that: $e^{\pi/4} < 1 + 4/\pi$.

I. *Solution by N. J. Kuenzi, Oshkosh, Wisconsin.*

Consider the function f defined on the set of positive real numbers by $f(x) = e^x - 1 - 1/x$. Note that f is differentiable on $(0, \infty)$ and that $f'(x) > 0$. It follows that f is strictly increasing and continuous on the set of positive reals. Since $f(1/2) < 0$ and $f(1) > 0$, it then follows from the Intermediate Value Theorem that there is a unique value c between 1/2 and 1 such that $f(c) = 0$. Newton's method yields .80646 as an approximate value of c. Since $\pi/4 < c$, it follows that $f(\pi/4) = e^{\pi/4} - 1 - 4/\pi < 0$.

II. *Solutions by Lawrence A. Ringenberg, Eastern Illinois University.*

1. Using published tables we find that $\pi/4 = 0.7853982$, $4/\pi = 1.2732395$, exp 0.786 = 2.194600. Therefore $\exp \pi/4 < 2.20 < 2.27 < 1 + 4/\pi$.

2. Comparing the perimeters of a unit circle and of a circumscribed regular 20-gon of that circle, we find that $\pi < 20 \tan 9° < 3.19$. For convenience, let $\pi/4 = p$. Then $p < 0.8$ and the given inequality is true if and only if $p(e^p - 1) < 1$. But

$$p(e^p - 1) = p \sum_{n=1}^{\infty} \frac{p^n}{n!} < p^2 + \frac{p^3}{2} + \frac{p^4}{6} \sum_{n=0}^{\infty} \left(\frac{p}{4}\right)^n < 0.64 + 0.26 + 0.07(1.25) < 0.99.$$

Therefore the proof is complete.

Also solved by Charusheel N. Bapat, Poona, India; Leon Bankoff, Los Angeles, California; Martin Berman, Bronx Community College, New York; Maxey Brooke, Sweeny, Texas; S. R. Conrad, Bayside, New York; Romae J. Cormier, Northern Illinois University; Clayton W. Dodge, University of Maine at Orono; Edward T. Frankel, Schenectady, New York; Ralph Garfield, College of Insurance, New York; Vaclav Konecny, Jarvis Christian College, Texas; Lew Kowarski, Morgan State College, Maryland; Paul Lavoie, College de Sherbrooke, Quebec, Canada; Peter A. Lindstrom, Genesee Community College, New York; Christopher L. Morgan and Jack S. Zelver (jointly), California State University at Hayward; A. J. Patschke, Rock Island, Illinois; Robert S. Stacy, Albuquerque, New Mexico; E. F. Schmeichel, Itasca, Illinois; James Tattersall, Attleboro, Massachusetts; K. L. Yocom, South Dakota State University; Wolf R. Umbach, Rottdorf, Germany; and the proposer.

A Corrected Equality

848. [November, 1972] *Proposed by J. Prasad, University of Dar Es Salaam, Tanzania.*

Prove that:

$$\sum_{s=0}^{n+1} (r + s)!/s! = \binom{n + r}{n - r} r!,$$

$r = 1, 2, 3 \cdots n$.

Solution by Lucille D. Roinestad, Secretary, Senior Mathematics Seminar Class, Fort Lewis College, Colorado.

Here is a solution to the corrected Problem 848 submitted by the Senior Math Seminar Class of Fort Lewis College.

$$\sum_{s=0}^{n+1} \frac{(r + s)!}{s!} = r! \sum_{s=0}^{n+1} \binom{r + s}{r} = \binom{n + r + 2}{n + 1} r!,$$

using the well-known fact that

$$\sum_{s=0}^{n+1}\binom{r+s}{r}=\binom{n+r+2}{n+1},$$

which is easily shown by induction.

Also solved by M. T. Bird, California State University, San Jose; Gladwin E. Bartel, Otero Junior College, Colorado; Clayton W. Dodge, University of Maine at Orono; Ralph Garfield, College of Insurance, New York; Richard A. Gibbs, Fort Lewis College, Colorado; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; John M. Howell, Littlerock, California; Vaclav Konecny, Jarvis Christian College, Texas; N. J. Kuenzi, University of Wisconsin at Oshkosh; D. L. Muench, St. John Fisher College, New York; Robert S. Stacy, Albuqerque, New Mexico; E. F. Schmeichel, Itasca, Illinois; Wolf R. Umbach, Rottdorf, Germany; and the proposer.

A Toroidal Surface

849. [November, 1972] *Proposed by William Wernick, City College of New York.*

If PB is a line perpendicular to the plane of the triangle ABC, find the necessary and sufficient conditions that angle $APC <$ angle ABC.

Solution by Michael Goldberg, Washington, D.C.

Draw the circumscribing circle of the triangle ABC. Rotate the arc ABC about the line AC to produce a toroidal surface. Then, every point Q on this toroidal surface satisfies the equation angle $AQC =$ angle ABC. Hence, a necessary and sufficient condition that angle APC be less than angle ABC is that P lie outside this toroidal surface.

If the perpendicular from B to the line AC falls outside the interval AC, then the line BP meets the toroidal surface at points Q_1 and Q_2 which are equidistant from B. Then point P must be taken beyond Q_1 or Q_2.

If the perpendicular from B to line AC falls inside the interval AC, then the points Q_1 and Q_2 coincide with B. Any point P on line BP, other than B, will suffice.

Also solved by Charusheel N. Bapat, Poona, India; Clayton W. Dodge, University of Maine at Orono; M. G. Greening, University of New South Wales, Australia; Lew Kowarski, Morgan State College, Maryland; Lawrence A. Ringenberg, Eastern Illinois University; Vaclav Konecny, Jarvis Christian College, Texas; and the proposer.

Condition for Convergence

850. [November, 1972] *Proposed by Richard Dykstra, University of Missouri-Columbia.*

If $a_0 > 0$, $0 < \alpha < 1$, and $a_i = a_{i-1} + a_{i-1}^{\alpha}$, $i = 1,2,3,\cdots$, for what values of β will $\sum_{k=1}^{n}(1/a_k)^{\beta}$ converge?

Solution by Ralph Garfield, College of Insurance, New York.

Clearly the upper limit of the summation is ∞. Applying the ratio test $\sum_{k=1}^{\infty}(1/a_k)^{\beta}$

converges if for n large enough $\left(\frac{a_n}{a_{n+1}}\right)^\beta < 1$. We can forget the absolute value signs since $a_0 > 0$. But $a_{n+1} = a_n + a_n^\alpha$

$$\therefore \left(\frac{a_n}{a_{n+1}}\right)^\beta = \left(\frac{a_n}{a_n + a_n^\alpha}\right)^\beta = \left(\frac{1}{1 + a_n^{\alpha-1}}\right)^\beta.$$

Since $a_0 > 0 \Rightarrow a_n > 0$ for all n and

$$0 < \alpha < 1 \Rightarrow -1 < \alpha - 1 < 0 \Rightarrow$$

$$1 < (1 + a_n^{\alpha-1}) < 2 \Rightarrow \left(\frac{1}{1 + a_n^{\alpha-1}}\right)^\beta < 1 \text{ if } \beta > 0,$$

$\therefore$ required condition is $\beta > 0$.

Also solved by Merrill Barnebey, University of Wisconsin at LaCrosse; N. J. Kuenzi, University of Wisconsin at Oshkosh; Paul Lavoie, College de Sherbrooke, Quebec, Canada; Robert S. Stacy, Albuquerque, New Mexico; and the proposer.

A Prime Divisor

851. [November, 1972] *Proposed by D. Rameswar Rao, Sitafalmandi, Secunderabad, India.*

If $p = 2k + 1$ is a prime number and p does not divide $a - 1$, a or $a + 1$, then show that p divides $(a^{2k} - 1)/(a^2 - 1)$.

I. *Solution by G. A. Heuer, Concordia College, and Karl Heuer, Student at Moorhead High School (jointly).*

By Fermat's Little Theorem, $a^{p-1} = (a-1)^{p-1} = (a+1)^{p-1} = 1$ in the field $Z/(p)$. It follows that $[(a^{2k} - 1)/(a^2 - 1)]^{p-1} = 0$, and hence that $(a^{2k} - 1)/(a^2 - 1) = 0$, in $Z/(p)$.

II. *Generalization by G. A. Heuer, Concordia College.*

We prove the following generalization: Let r be an odd integer greater than 3 which is relatively prime to $a - 1$, a, and a $+ 1$. Then r divides $(a^{\Phi(r)} - 1)/(a^2 - 1)$. For, by Euler's generalization of Fermat's theorem, $a^{\Phi(r)} \equiv (a-1)^{\Phi(r)} \equiv (a+1)^{\Phi(r)} \equiv 1 \pmod r$, and therefore $(a^2 - 1)^{\Phi(r)} \equiv 1 \pmod r$. Thus $a^2 - 1$ is invertible in the ring $Z/(r)$, and $(a^{\Phi(r)} - 1)/(a^2 - 1) = (a^{\Phi(r)} - 1)(a^2 - 1)^{\Phi(r)-1} = 0$ in $Z/(r)$.

Also solved by J. D. Baum, Oberlin College, Ohio; Stephen D. Brown, Southern Colorado State College; Clayton W. Dodge, University of Maine at Orono; Richard A. Gibbs, Fort Lewis College, Colorado; M. G. Greening, University of New South Wales, Australia; Erwin Just, Bronx Community College, New York; Richard Kerns, Hamburg, Germany; Lew Kowarski, Morgan State College, Maryland; Arthur Marshall, Madison, Wisconsin; Raymond A. Maruca, Delaware County Community College, Pennsylvania; Bob Prielipp, University of Wisconsin at Oshkosh; Sally Ringland, Clarion State College, Pennsylvania; E. F. Schmeichel, Itasca, Illinois; Robert S. Stacy, Albuquerque, *New Mexico; James Tattersall, Attleboro, Massachusetts; Dmitri Thara, California State University, San Jose; Charles W. Trigg, San Diego, California; Wolf R. Umbach, Rottdorf, Germany; Kenneth M. Wilke, Topeka, Kansas; Quazi Zameeruddin, K. N. College, Delhi, India; and the proposer.*

Comment on Problem 84

84. [January, 1951, September, 1965, September, 1966, September, 1971] *Proposed by Dewey Duncan, Los Angeles, California.*

We define a heterosquare as a square array of the first n^2 positive integers, so arranged that no two rows, columns, and diagonals (broken as well as straight) have the same sum.

(a) Show that no heterosquare of order two exists.

(b) Find a heterosquare of order three.

Comment by Charles W. Trigg, San Diego, California.

On Page 237, of the September, 1971, issue of this MAGAZINE, the square in the upper *left* hand corner of the table is a Nagara almost heterosquare, not the one "in the upper right hand corner" as stated.

Duncan's original proposal was numbered "86" not "84".

Comment on Problem 803

803. [September, 1971, and May, 1972] *Proposed by Kenneth Rosen, University of Michigan.*

Let x and y be positive real numbers with $x + y = 1$. Prove that $x^x + y^y \geqq \sqrt{2}$ and discuss conditions for equality.

Comment by Andrzej Makowski, Warsaw, Poland.

The solution of Problem 803 given by Leon Bankoff contains a gap. It is not clear why it is asserted that if $(x, y) \neq (\frac{1}{2}, \frac{1}{2})$, then $x^x + y^y > \sqrt{2}$. It is evident that for $(x, y) \neq (\frac{1}{2}, \frac{1}{2})$ we have $x^x + y^y > 2\sqrt{x^x y^y}$, but why $x^x + y^y > \sqrt{2}$?

Comment on Problem 829

829. [March, 1972, and January, 1973] *Proposed by John D. Baum, Oberlin College.*

It is well known that a positive integer can be written as the sum of consecutive integers if and only if it is not a power of two. If a positive integer is so expressible, its representation is not necessarily unique. For example,

$$15 = 7 + 8 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5.$$

For integers of what form are their expressions as sums of positive consecutive integers unique?

Comment by L. Earl Bush, Kent, Ohio.

The solution to this problem is given in the third corollary to Theorem 4 of my paper, *On the expression of an integer as the sum of an arithmetic series*, The AMERICAN MATHEMATICAL MONTHLY, 5, 37 (1930) p. 356. However, I give credit to T. E. Mason for this result, given in The AMERICAN MATHEMATICAL MONTHLY, 5, 19 (1912) p. 46.

Comment on Problem 830

830. [March, 1972, and January, 1973] *Proposed by Frank Dapkus, Seton Hall University.*

Find a right triangle with the smallest area that can be partitioned into two triangles with all integral sides.

Comment by Charles W. Trigg, San Diego, California.

The right triangle with the smallest area, 54, that can be partitioned into two triangles with integral sides *by a line drawn from the vertex of an acute angle* is the triangle (9, 12, 15). The partitions are the primitive Pythagorean triangle (5, 12, 13) and the primitive Heronian triangle (4, 13, 15), which have areas of 30 and 24, respectively.

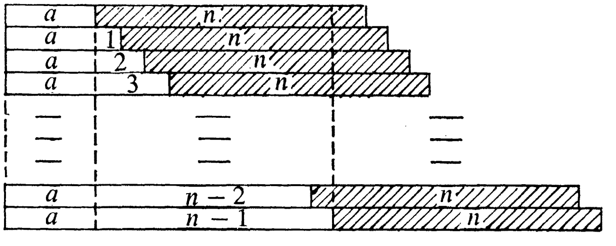
Comment on Q555

Q555. [November, 1972] Show that the sum of n consecutive integers plus n^2 is equal to the sum of the next n consecutive integers.

[*Submitted by Charles W. Trigg*]

Alternate solution by William Wernick, City College of New York.

Here is a nice geometric way of establishing the relationship. In the n layers, the n consecutive integers are the clear rectangles, the shaded rectangles add n^2, and the complete figure is the sum of the next n consecutive integers.



Generalization by Charles W. Trigg, San Diego, California.

Q555 can be generalized to read: Show that the sum of n consecutive terms of an arithmetic progression with common difference, d, plus n^2d is equal to the sum of the next n consecutive terms of the progression. Now

$$\begin{array}{ccccccc} (a+d) & + & (a+2d) & + & (a+3d) & + \cdots + & (a+nd) \\ nd & + & nd & + & nd & + \cdots + & nd \\ \hline [a+(n+1)d] & + & [a+(n+2)d] & + & [a+(n+3)d] & + \cdots + & [a+2nd]. \end{array}$$

Since the second line of the addition contains n terms, the proof is complete.

ANSWERS

A573. The larger that k becomes, the closer the perimeter (and hence the area) is to twice the length. Hence the width approaches 2 as a limit.

A574. Let $F(x) = C^2(x) + S^2(x)$. Clearly $C'(x) = -S(x)$ and $S'(x) = C(x)$ whence $F'(x) = 2C(x)C'(x) + 2S(x)S'(x) = 0$. Thus $F(x)$ is a constant. But $F(0) = 1$ whence $C^2(x) + S^2(x) = 1$ for all x.

A575. The entire time difference of 2/3 hours occurred in the 50 miles beyond the point of the original breakdown. Letting S denote the original speed and T the original time to cover these 50 miles, we have $T = \dfrac{50}{3/5S} = \dfrac{50}{S} + 2/3$ or $S = 50$ mph.

A576. Let the tetrahedron be given by the n coordinate planes of an n-dimensional rectangular coordinate system $x_1, x_2 \cdots x_n$ and the hyperplane P: $\sum x_i/a_i = 1$. If V and B denote the contents of the simplex and the face opposite O (the origin), repectively, then $V = dB/n$ where d denotes the distance from O to P. Since $V = \dfrac{1}{n!}\,\pi a_2$ and $P = \{\sum a_i^{-2}\}^{-1/2}$

$$B^2 = \{(n-1)!\}^{-2}\{\sum a_i^{-2}\}\pi a_i^2.$$

Now each term in the summation for B^2 corresponds to the square of the content of the remaining faces (the $(n-1)$-dimensional version of the expression for V). The case $n = 2$ corresponds to the Pythagorean theorem but we have not proved it since it was used implicitly in the solution. However, the result does generalize the known result for the tetrahedron ($n = 3$).

A577. Let $\{p_1, p_2 \cdots p_n\}$ be an arbitrary set of n distinct primes, and set $f(x) = x + \prod_{i=1}^{n}(x - p_i)$. It is readily seen that $f(x)$ may be written in the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

in which the a_i are integers, an $a_n \neq 0$, and for each i, $f(p_i) = p_i$, so the range of f contains $\{p_1, p_2 \cdots p_n\}$.

(Quickies on pages 231–232.)

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